ECE 486 Important Information for the Final

The final exam will take place

7:00 – 10:00 p.m., Thursday, May 7, 1015 ECE Building

The test will be *comprehensive*, but it will emphasize frequency domain (§6.1-6.7) and state space (§7.1-7.10) techniques. In a nutshell, I expect you to know:

- General models: State space; transfer function; block diagrams.
- Linearization.
- Transient and steady-state response: DC gain; Final Value Theorem.
- Second-order response and the effect of poles and zeros; time-domain specifications (rise time, overshoot, peak time, settling time) and their relation to pole locations.
- Stability: definition; necessary condition for stability; Routh–Hurwitz criterion; necessary and sufficient conditions for 2nd- and 3rd-order polynomials.
- Open-loop and closed-loop feedback control: reference-to-output and reference-to-error transfer functions; tracking error.
- Simple compensators: PID, lead, lag; effect of controller parameters on time-domain specs and on steady-state response.
- Root locus methods as developed in class (Rules A—F): Evans' canonical form; phase condition; effect of PD/lead and PI/lag compensation on the root locus.
- Frequency domain basics: Bode plots, Nyquist plots, how to sketch them, how to relate them for a given transfer function.
- Bode's gain-phase relationship.
- Frequency domain design: Crossover frequency; bandwidth; phase and gain margins; PD/lead and PI/lag compensation; choosing lead/lag parameters to satisfy given specs (bandwidth, PM/GM, steady-state tracking errors).
- The Nyquist Stability Criterion: $N = Z - P$; reading stability ranges given the Nyquist plot and knowledge of open-loop poles and zeros.
- Reading stability margins (PM and GM) off a Nyquist plot.
- State-space realizations of transfer functions:

$$\dot{x} = Ax + Bu,\ y = Cx \qquad \Longrightarrow \qquad Y(s) = C(Is - A)^{-1}BU(s)$$

- Canonical forms: CCF and OCF.
- Controllability and observability criteria for SISO systems.
- Coordinate transformations: effect on transfer function, characteristic polynomial, controllability and observability matrices; converting a given controllable (respectively, observable) system to CCF (respectively, OCF) and back.
- Closed-loop pole assignment by full-state feedback: $u = -Kx$.
- Observer design: $\dot{\widehat{x}} = (A - LC)\widehat{x} + Ly + Bu$.
- Dynamic output feedback: $u = -K\widehat{x}$.
- The Separation Principle.

The test will be closed-book, no calculators allowed. You can bring two double-sided sheets of notes. The bare minimum of the material you need to know will be attached to the exam and reproduced below. *However,* you are responsible for all of the content outlined above. The exam will be written in the same style as the midterms (In fact, I will pick one problem "at random" from each of the two midterms). The length will be less than two midterms.

If you understand all of the homework and laboratory concepts, then you are certain to receive an 'A' on the final.

I wish you good luck on the exam, and a happy future!

— ○ —

Useful Facts

Unilateral Laplace transforms:

$$f(t),\ t \geq 0 \quad \xrightarrow{\mathscr{L}} \quad F(s) = \int_0^\infty f(t)e^{-st}\mathrm{d}t,\ s \in \mathbb{C}$$

$$\mathscr{L}\left[f'(t)\right] = sF(s) - f(0)$$

$$\mathscr{L}\left[f''(t)\right] = s^2F(s) - sf(0) - f'(0)$$

Second-order system:

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s+\sigma)^2 + \omega_d^2} \qquad \omega_n, \zeta > 0$$

Rise time: $t_r \approx \dfrac{1.8}{\omega_n}$

Peak time: $t_p = \dfrac{\pi}{\omega_n\sqrt{1-\zeta^2}}$

Overshoot: $M_p = \exp\left(-\dfrac{\pi\zeta}{\sqrt{1-\zeta^2}}\right)$

Settling time: $t_s^{5\%} \approx \dfrac{3}{\zeta\omega_n}$

Stability criteria for polynomials:

- a monic polynomial $p(s) = s^n + a_1 s^{n-1} + \ldots + a_{n-1}s + a_n$ is *stable* if all of its roots are in the open LHP
- 2nd-order polynomial

  $$p(s) = s^2 + a_1 s + a_2$$

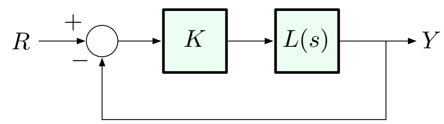
  is stable if and only if $a_1, a_2 > 0$
- 3rd-order polynomial

  $$p(s) = s^3 + a_1 s^2 + a_2 s + a_3$$

  is stable if and only if $a_1, a_2, a_3 > 0$ and $a_1 a_2 > a_3$

Root locus Let L be a proper transfer function of the form

$$L(s) = \frac{b(s)}{a(s)} = \frac{s^m + b_1 s^{m-1} + \ldots + b_{m-1}s + b_m}{s^n + a_1 s^{n-1} + \ldots + a_{n-1}s + a_n}$$



The *root locus* is the set of all $s \in \mathbb{C}$ such that

$$1 + KL(s) = 0 \qquad \Longleftrightarrow \qquad a(s) + Kb(s) = 0$$

Phase condition: a point $s \in \mathbb{C}$ is on the RL if and only if

$$\angle L(s) = \angle \frac{b(s)}{a(s)} = \angle \frac{(s - z_1)\ldots(s - z_m)}{(s - p_1)\ldots(s - p_n)} = 180^\circ \mod 360^\circ$$

Rules for sketching root loci

- Rule A: n branches ($n = \#$(open-loop poles))
- Rule B: branches start at open-loop poles $p_1, \ldots, p_n$
- Rule C: m of the branches end at open-loop zeros $z_1, \ldots, z_m$ (L is proper: $m \leq n$)
- Rule D: a point $s \in \mathbb{R}$ is on the RL if and only if there is an *odd* number of *real* open-loop poles and zeros to the right of it
- Rule E: if $n - m > 0$, the remaining $n - m$ branches approach ∞ along asymptotes departing from the point

  $$\alpha = \frac{\sum_{i=1}^n p_i - \sum_{j=1}^m z_j}{n - m}$$

  at angles

  $$\frac{(2\ell + 1) \cdot 180^\circ}{n - m}, \qquad \ell = 0, 1, \ldots, n - m - 1.$$

- Rule F: $j\omega$-crossings
  - find the critical value(s) of K (if any) that will make the characteristic polynomial $a(s) + Kb(s)$ unstable
  - for each of these critical values, solve

    $$a(j\omega) + Kb(j\omega) = 0$$

    for critical frequencies ω

Bode plots A transfer function $G(j\omega)$ is in *Bode form* if it is written as a product of (some or all of) the following three types of factors:

- Type 1 — nth-order zero or pole at the origin, $K_0(j\omega)^n$, $K_0 > 0$, n is an integer
- Type 2 — real zero or pole, $(j\omega\tau + 1)^{\pm 1}$, $\tau > 0$
- Type 3 — complex zero or pole, $\left[\left(\frac{j\omega}{\omega_n}\right)^2 + 2\zeta\frac{j\omega}{\omega_n} + 1\right]^{\pm 1}$, $\quad \omega_n > 0$, $0 < \zeta < 1$

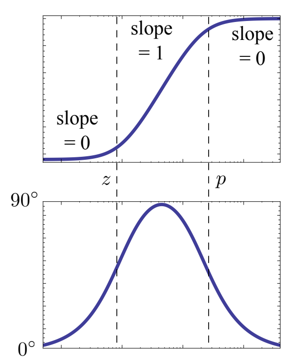
Magnitude and phase relationships:

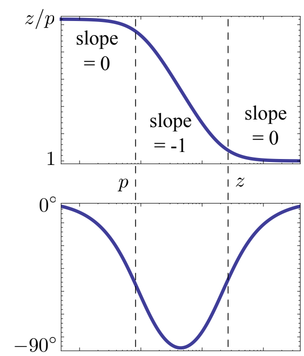
	low frequency	real zero/pole	complex zero/pole
magnitude slope	n	up/down by 1	up/down by 2
phase	$n \times 90°$	up/down by 90°	up/down by 180°

Crossover frequency: $|G(j\omega_c)| = 1$

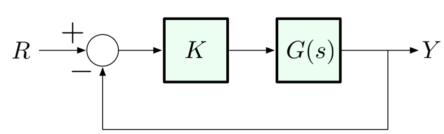
Bode plots for lead and lag compensators

$$\text{lead: } D(s) = K\frac{s/z+1}{s/p+1}, \qquad z < p \qquad \text{lag: } D(s) = \frac{s+z}{s+p}, \qquad z > p$$





Stability margins — assume K is stabilizing



- **Gain Margin (GM)**: the factor by which K has to be multiplied for the closed-loop system to become unstable
- **Phase Margin (PM)**: the amount by which the phase of $G(j\omega_c)$ differs from $\pm 180°$ (the sign depends on the magnitude slope of the Bode plot of KG)

Nyquist plots For a transfer function $H(s)$, the Nyquist plot is the set of all points

$$\Big(\text{Re } H(j\omega), \text{Im } H(j\omega)\Big), \qquad -\infty < \omega < \infty$$

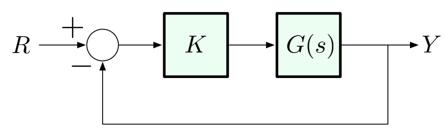
The Argument Principle

$$N = Z - P,$$

where:

- $N = \#(\circlearrowright$ of 0 by the Nyquist plot of $H)$
- $Z = \#(\text{RHP zeros of } H)$
- $P = \#(\text{RHP poles of } H)$

Nyquist Stability Criterion — consider the unity feedback configuration:



Then

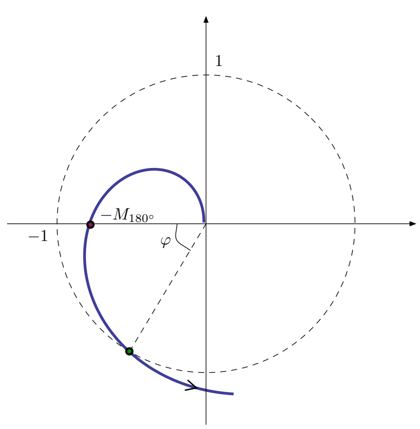
$$N = Z - P,$$

where:

- $N = \#(\circlearrowright$ of $-1/K$ by the Nyquist plot of $G)$
- $Z = \#(\text{RHP closed-loop poles})$
- $P = \#(\text{RHP open-loop poles})$

Stability margins from Nyquist plots

$$\text{GM} = \frac{1}{M_{180^\circ}}, \qquad \text{PM} = \varphi$$



State-space models:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \qquad x \in \mathbb{R}^n,\, u \in \mathbb{R}^m,\, y \in \mathbb{R}^p \qquad \longrightarrow \qquad Y(s) = C(Is - A)^{-1}BU(s) \end{aligned}$$

Coordinate transformations: if $T \in \mathbb{R}^{n \times n}$ is an invertible matrix, then, for $\bar{x} = Tx$,

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u, \qquad y = \bar{C}\bar{x}$$

where $\bar{A} = TAT^{-1}$, $\bar{B} = TB$, $\bar{C} = CT^{-1}$.

Controllability: a single-input system with state-space realization $\dot{x} = Ax + Bu$, $y = Cx$ is controllable if the $n \times n$ controllability matrix

$$\mathcal{C}(A,B) = \left[B \,|\, AB \,|\, \ldots \,|\, A^{n-1}B \right]$$

is invertible.

Controller canonical form: a state-space model is in Controller Canonical Form (CCF) if

$$A = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \ldots & -a_2 & -a_1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

Key facts:

- a system in CCF is always controllable
- $\det(Is - A) = s^n + a_1 s^{n-1} + \ldots + a_{n-1}s + a_n$

Observability: a single-output system with state-space realization $\dot{x} = Ax + Bu$, $y = Cx$ is observable if the $n \times n$ observability matrix

$$\mathcal{O}(A,C) = \begin{bmatrix} C \\ \hline CA \\ \hline \vdots \\ \hline CA^{n-1} \end{bmatrix}$$

is invertible.

Observer canonical form: a state-space model is in Observer Canonical Form (OCF) if

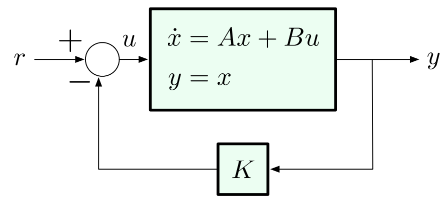
$$A = \begin{pmatrix} 0 & 0 & \ldots & 0 & 0 & -a_n \\ 1 & 0 & \ldots & 0 & 0 & -a_{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 & -a_2 \\ 0 & 0 & \ldots & 0 & 1 & -a_1 \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & 0 & \ldots & 0 & 1 \end{pmatrix}$$

Key facts:

- a system in OCF is always observable
- $\det(Is - A) = s^n + a_1 s^{n-1} + \ldots + a_{n-1}s + a_n$

Full-state feedback control:

$$\begin{aligned} \text{Plant:} \quad & \dot{x} = Ax + Bu \\ & y = x \\ \text{Controller:} \quad & u = -Kx + r \qquad\qquad (r = \text{reference input}) \end{aligned}$$



Transfer function from R to Y:

$$Y(s) = (Is - A + BK)^{-1} BR(s)$$

If the pair (A, B) is controllable, then controller poles (eigenvalues of $A - BK$ = roots of $\det(Is - A + BK)$) may be assigned arbitrarily by an appropriate choice of K.

Observer design:

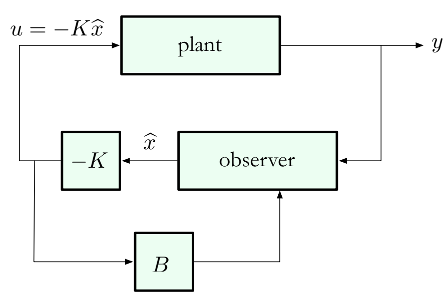
$$\begin{aligned} \text{Plant:} \quad & \dot{x} = Ax + Bu \\ & y = Cx \\ \text{Observer:} \quad & \dot{\widehat{x}} = (A - LC)\widehat{x} + Ly + Bu \\ \text{Controller:} \quad & u = -K\widehat{x} \end{aligned}$$

If the pair (A, C) is observable, then observer poles (eigenvalues of $A - LC$ = roots of $\det(Is - A + LC)$) may be assigned arbitrarily by an appropriate choice of L.

The overall observer-controller system is:

$$\begin{aligned} \dot{\widehat{x}} &= (A - LC)\widehat{x} + Ly + B \underbrace{(-K\widehat{x})}_{=u} \\ &= (A - LC - BK)\widehat{x} + Ly \\ u &= -K\widehat{x} \qquad\qquad \text{(dynamic output feedback)} \end{aligned}$$

— this is a dynamical system with input y and output u



Transfer function of the overall dynamic-output feedback controller:

$$U(s) = -K(Is - A + LC + BK)^{-1} LY(s)$$