University of Illinois Spring 2005

ECE 413: Problem Set 8

Due:	Wednesday March 16 at the beginning of class.
Reading:	Ross, Chapters 4 and 5
Noncredit exercises:	Ross, Chapter 5, Problems 1-3, 5-8; Theoretical Exercises 1, 8.
Reminders:	No class on Friday March 11 on account of EOH
	No class on Friday March 18 (time off for first evening hour exam)
	No class on Monday March 28 (time off for second evening hour exam)

This Problem Set contains six problems

1. The dice game of *craps* begins with the player (called the *shooter*) rolling two fair dice. If the sum is 2, 3, or 12, the shooter loses the game. If the sum is a 7 or 11, the shooter wins the game. If the sum is any of 4, 5, 6, 8, 9, 10, then the shooter has neither won nor lost (as yet). The number rolled is called the shooter's *point*, and what happens next is described in parts (b) and (c) below.

(**a**) What is the probability that the shooter wins the game on the first roll? What is the probability that the shooter loses the game on the first roll? What is the probability that the shooter's point is $i, i \in \{4,5,6,8,9,10\}$? I need six answers here, folks!

(**b**) Suppose that the shooter's point is i. The shooter rolls the dice again. If the result is i, the shooter is said to have *made the point* and wins the game. If the result is 7, the shooter loses the game (craps out). If the result is anything else, the shooter rolls the dice again. This continues until the shooter either makes the point or craps out. For each $i \in \{4,5,6,8,9,10\}$, compute the probability that the shooter wins the game. Note that these are *conditional* probabilities of winning given that the shooter's point is i.

(**c**) Conditioned on the shooter's point being i, what is the expected number of dice rolls till the game ends? (Note: one dice roll = rolling two dice simultaneously). What is the expected number of dice rolls in a game of craps?

(**d**) If the shooter's point is 8, then side-bets are offered at 10 to 1 odds that the shooter will make the point *the hard way* by rolling $(4,4)$. Is this a fair bet? (Remember that 10 to 1 odds means if you bet a dollar, you will get ten dollars back (in addition to your original dollar back)).

2. Which of the following functions $F(u)$ are valid CDFs? For those that are valid CDFs, compute the probability that the absolute value of the random variable exceeds 0.5.

(**a**) $F(u) = \begin{cases} 0 & u < 0, \\ u^2, & 0 \le u < 1, \\ 1, & u \ge 1. \end{cases}$ (**b**) $F(u) = \begin{cases} 0 & u < 1, \\ 2u - u^2, & 1 \le u \le 2, \\ 1, & u > 2. \end{cases}$

(**c**) $F(u) = \begin{cases} \frac{1}{2}\exp(2u) & u \le 0, \\ 1 - \frac{1}{4}\exp(-3u), & u > 0, \end{cases}$ (**d**) $F(u) = \begin{cases} \frac{1}{2}\exp(2u) & u < 0, \\ 1 - \frac{1}{4}\exp(-3u), & u \ge 0, \end{cases}$

3. The number of hours that a student spends on ECE 440 homework is a random variable $\mathcal{X}$ with CDF

$$F_{\mathcal{X}}(u) = \begin{cases} 0, & u < 0, \\ (1+u)/8, & 0 \le u < 1, \\ 1/2, & 1 \le u < 2, \\ (4+u)/8, & 2 \le u < 4, \\ 1, & u \ge 4. \end{cases}$$

Note that this is a *mixed* random variable: it takes on some values with nonzero probability (like a discrete random variable) but also takes on all values in intervals of the real line (like a continuous random variable).

(**a**) Find $P\{\mathcal{X} = 2\}$, $P\{\mathcal{X} < 2\}$, $P\{\mathcal{X} > 2\}$, $P\{1 \le \mathcal{X} \le 3\}$, and $P\{\mathcal{X} > 2 \mid \mathcal{X} > 0\}$.

(**b**) Find $\mathsf{E}[\mathcal{X}]$.

4. The expectation of a nonnegative random variable $\mathcal{X}$ is

$$\mathsf{E}[\mathcal{X}] = \int_0^\infty [1 - F_{\mathcal{X}}(u)]\, du = \int_0^\infty P\{\mathcal{X} > u\} du.$$

(**a**) Use this result to prove that if $\mathcal{X}$ is a discrete random variable that takes on nonnegative *integer* values, then $\mathsf{E}[\mathcal{X}] = \sum_{k=0}^{\infty} P\{X > k\} = \sum_{i=1}^{\infty} P\{X \geq i\}$. (cf. Theoretical Exercise 6, p. 180 of Ross).

(**b**) For $k = 0, 1, 2, \ldots$, find $P\{\mathcal{X} > k\}$ for a *geometric* random variable with parameter p. Use these results together with the result of part (a) to provide a different proof of the fact that $\mathsf{E}[\mathcal{X}] = p^{-1}$.

(**c**) Theoretical Exercise 7 on page 180 of Ross.

5. Nine functions $f(u)$ are shown below. Note that in each case, $f(u) = 0$ for all u not in the interval specified. In each case,

- determine whether $f(u)$ is a valid probability density function (pdf).
- If $f(u)$ is not a valid pdf, determine if there exists a constant C such that $C \cdot f(u)$ is a valid pdf.

(**a**) $f(u) = 2u, \quad 0 < u < 1.$
(**b**) $f(u) = |u|, \quad |u| < \frac{1}{2}$
(**c**) $f(u) = 1 - |u|, \quad |u| < 1,$
(**d**) $f(u) = \ln u, \quad 0 < u < 1.$ Hint: $\ln u$ can be integrated by parts.
(**e**) $f(u) = \ln u, \quad 0 < u < 2,$
(**f**) $f(u) = \frac{2}{3}(u - 1), \quad 0 < u < 3,$
(**g**) $f(u) = \exp(-2u), \quad u > 0.$
(**h**) $f(u) = 4\exp(-2u) - \exp(-u), \quad u > 0,$
(**i**) $f(u) = \exp(-|u|), \quad |u| < 1,$

6. Problem 4 on page 228 of Ross.