

ECE 313
Probability with Engineering Applications

Lecture 41 – Limit Theorems II

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Chebyshev's Inequality

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$P\{|X - \mu| \geq k\} \leq \frac{\sigma^2}{k^2}$

$P\{|X - \mu| \geq k\} = \int_{-\infty}^{\mu - k} f_X(u) du + \int_{\mu + k}^{\infty} f_X(u) du = \int_{-\infty}^{\infty} g(u) f_X(u) du$

$g(u) = \begin{cases} 1 & \text{if } u \leq \mu - k \text{ or } u \geq \mu + k \\ 0 & \text{else} \end{cases} \Rightarrow h(u) = \frac{(u - \mu)^2}{k^2} \geq g(u)$

$P\{|X - \mu| \geq k\} = \int_{-\infty}^{\infty} g(u) f_X(u) du \leq \int_{-\infty}^{\infty} \frac{(u - \mu)^2}{k^2} f_X(u) du = \frac{\sigma^2}{k^2}$

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Weak Law of Large Numbers

Weak Law of Large Numbers

Thm. Let X_1, X_2, \dots be a sequence of i.i.d. r.v.'s with $E\{X_i\} = \mu < \infty$. Then for any $k > 0$

$P\left\{\left|\frac{X_1 + X_2 + \dots + X_n}{n} - \mu\right| \geq k\right\} \rightarrow 0$ as $n \rightarrow \infty$

P.P. $Z = \frac{X_1 + \dots + X_n}{n}$, $\mu_Z = \frac{n\mu}{n} = \mu$, $\text{Var}(Z) = \frac{\sigma^2}{n}$

$P\{|Z - \mu_Z| \geq k\} \leq \frac{\sigma_Z^2}{k^2} = \left(\frac{\sigma^2}{n}\right) \cdot \frac{1}{k^2}$

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Chebyshev, WLLN, Rel. Freq.

Chebyshev, WLLN, Rel. Freq.

Consider n indep. trials of some experiment, prob of success on any trial is p , $X_i = 1$ if success on i th trial, else $X_i = 0$

$\mu_{X_i} = p$, $\text{Var}(X_i) = p(1-p)$ [$X_i \sim \text{Bern}(p)$]

$Z = X_1 + X_2 + \dots + X_n = \text{Total \# successes}$ | $E\{Z\} = np$

$Z/n = \text{rel. freq. of success}$ | $\text{Var}(Z/n) = \frac{p(1-p)}{n}$

$P\left\{\left|\frac{X_1 + \dots + X_n}{n} - \mu\right| \geq \alpha\right\} = P\left\{\left|\frac{Z}{n} - p\right| \geq \alpha\right\} \leq \frac{p(1-p)}{n\alpha^2}$

Prob. of rel. freq. differs from true prob. of success by more than α

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Central Limit Theorem

Central Limit Theorem

Thm. Let X_1, X_2, \dots be a sequence of i.i.d. r.v.'s with $E\{X_i\} = \mu$, $\text{Var}(X_i) = \sigma^2$ then

$Z = \frac{X_1 + X_2 + \dots + X_n - n\mu}{\sigma\sqrt{n}}$ tends to Gauss.

as $n \rightarrow \infty$, i.e.

$P\{a \leq Z \leq b\} = \frac{1}{\sigma\sqrt{n}} \int_a^b e^{-u^2/2} du = \Phi\left(\frac{b - n\mu}{\sigma\sqrt{n}}\right) - \Phi\left(\frac{a - n\mu}{\sigma\sqrt{n}}\right)$

$Y = \sum X_i \Rightarrow \mu_Y = n\mu$, $\text{Var}(Y) = n\sigma^2 \Rightarrow \sigma_Y = \sqrt{n}\sigma$

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Example

Example

X_i are i.i.d., $E\{X_i\} = \mu$, $\text{Var}(X_i) = \sigma^2, i=1, \dots, n$

Let $Y = \sum X_i$

$P\{a \leq Y \leq b\} = \Phi\left(\frac{b - n\mu}{\sigma\sqrt{n}}\right) - \Phi\left(\frac{a - n\mu}{\sigma\sqrt{n}}\right)$

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DeMoivre-Laplace

Let Y be binomial (n, p) , $E[Y] = np$
 $\text{Var}(Y) = np(1-p)$

$P\{a \leq Y \leq b\} \approx \Phi\left(\frac{b - np}{\sqrt{np(1-p)}}\right) - \Phi\left(\frac{a - np}{\sqrt{np(1-p)}}\right)$

Relate to CLT??

$Y = X_1 + X_2 + \dots + X_n$, each X_i Bernoulli, p .
 $E[X_i] = p$, $\text{Var}(X_i) = p(1-p)$

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