Public-Key Cryptography

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Lecture 6

Public-Key Encryption

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Diffie-Hellman Key-Exchange, El Gamal Encryption

PKE scheme

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- SKE:
  - Syntax
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  - Correctness
    - $\forall K \in$ Range(KeyGen),
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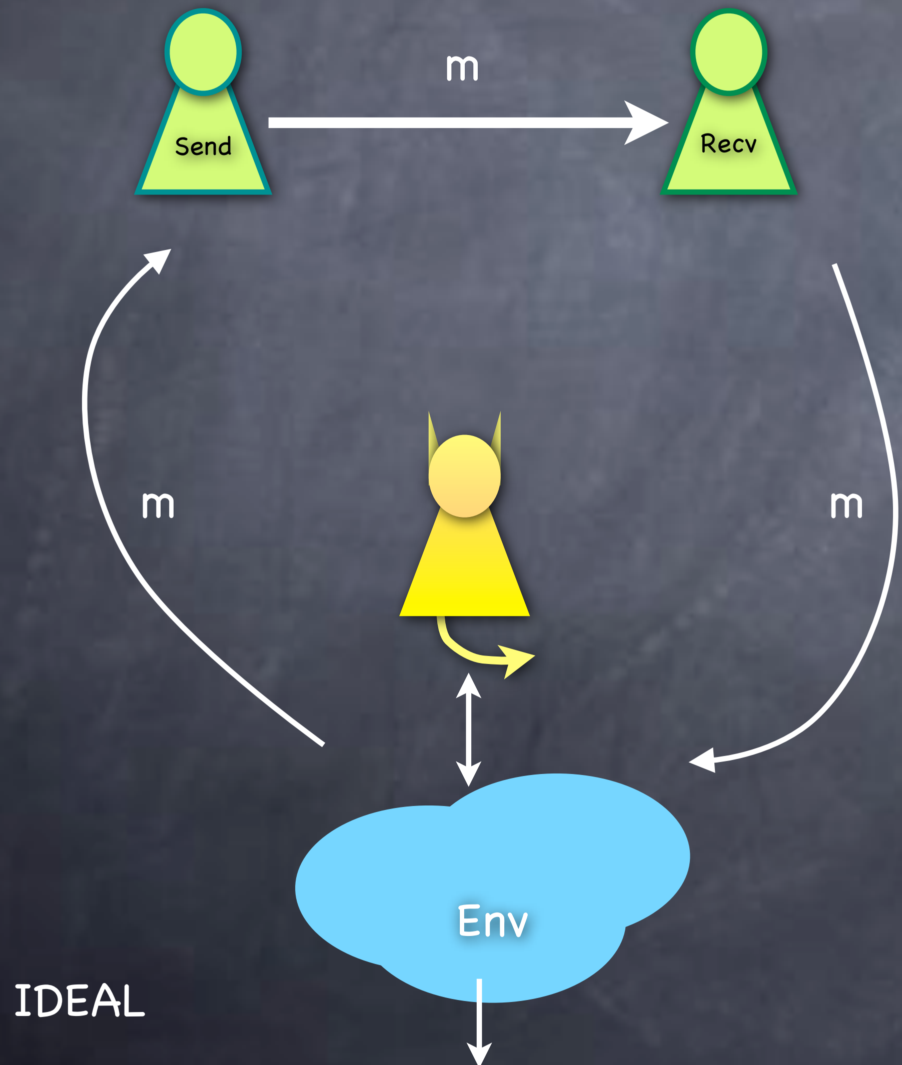
  - Security (IND-CPA, PKE version)

SIM-CPA (PKE Version)



Secure (and correct) if:

∀

∃ s.t.

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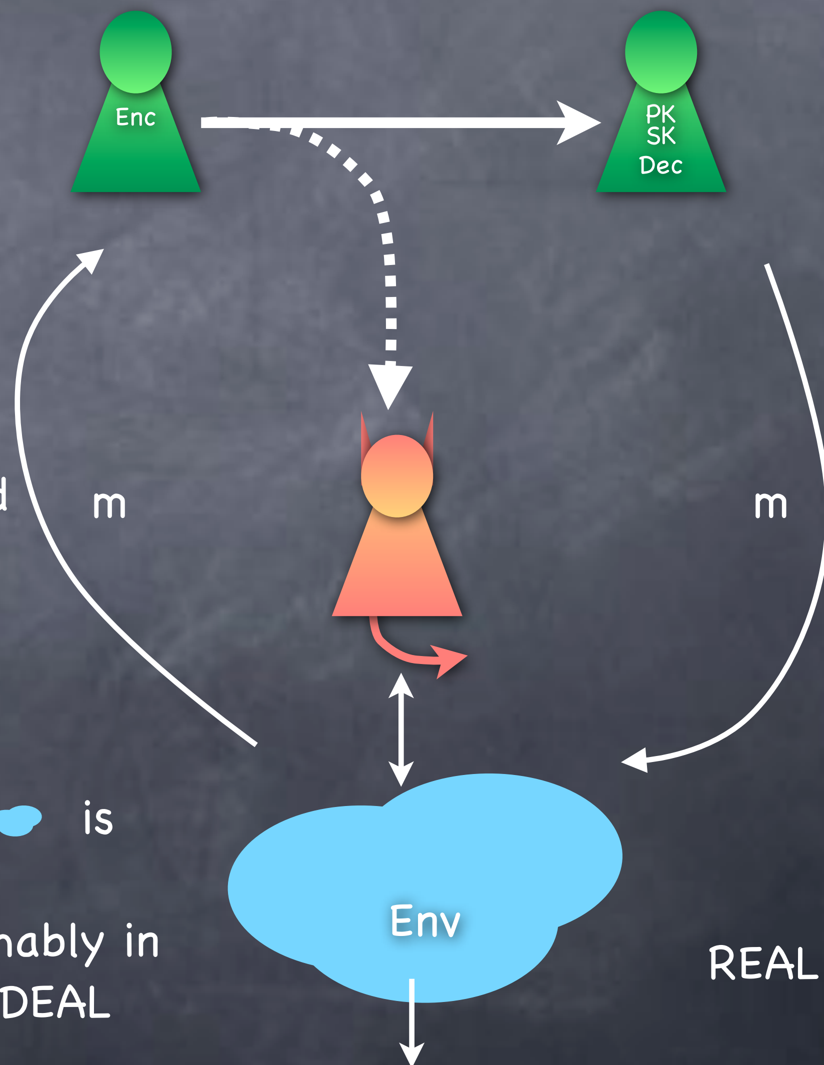
output of is distributed indistinguishably in REAL and IDEAL



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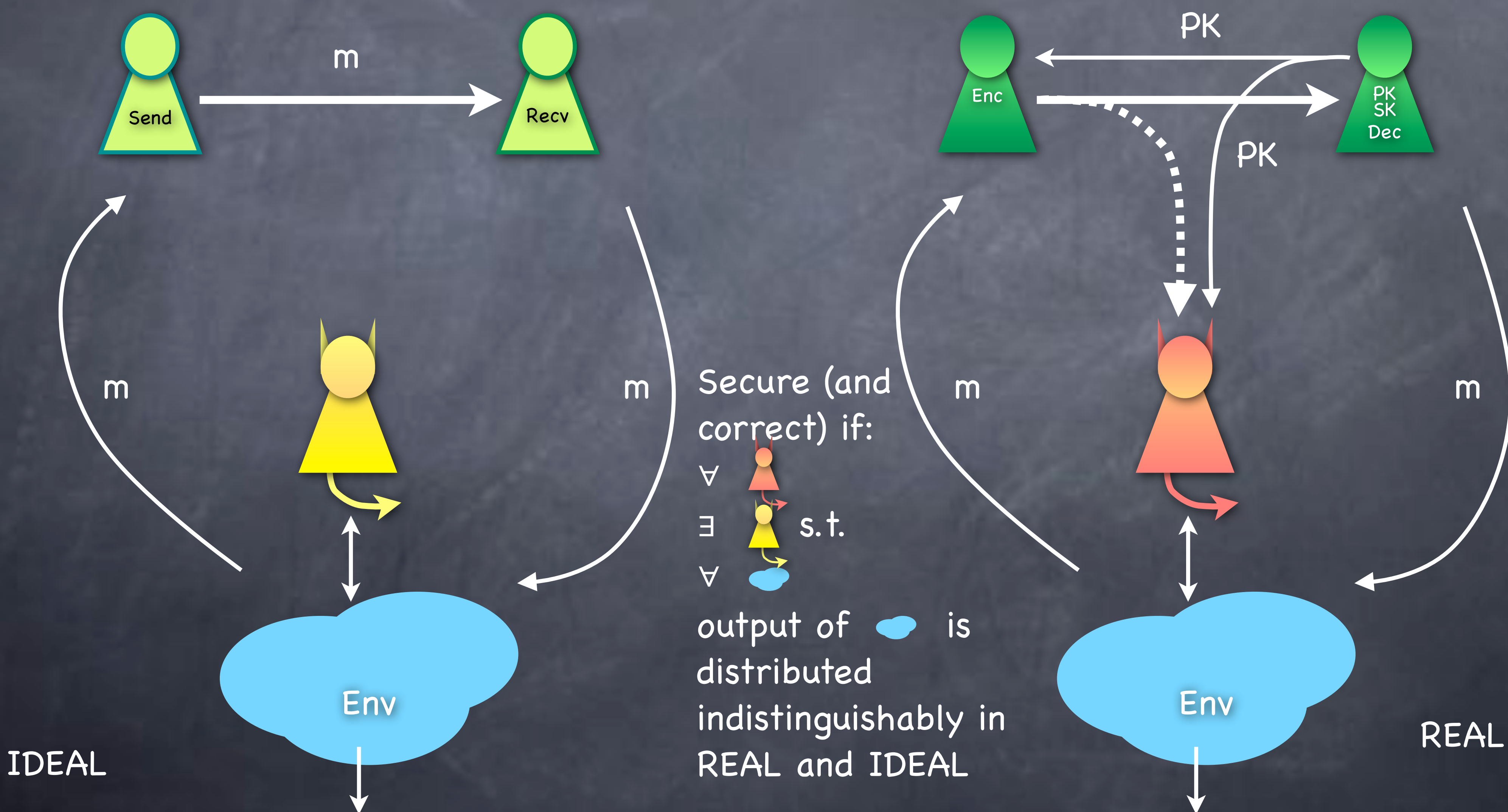

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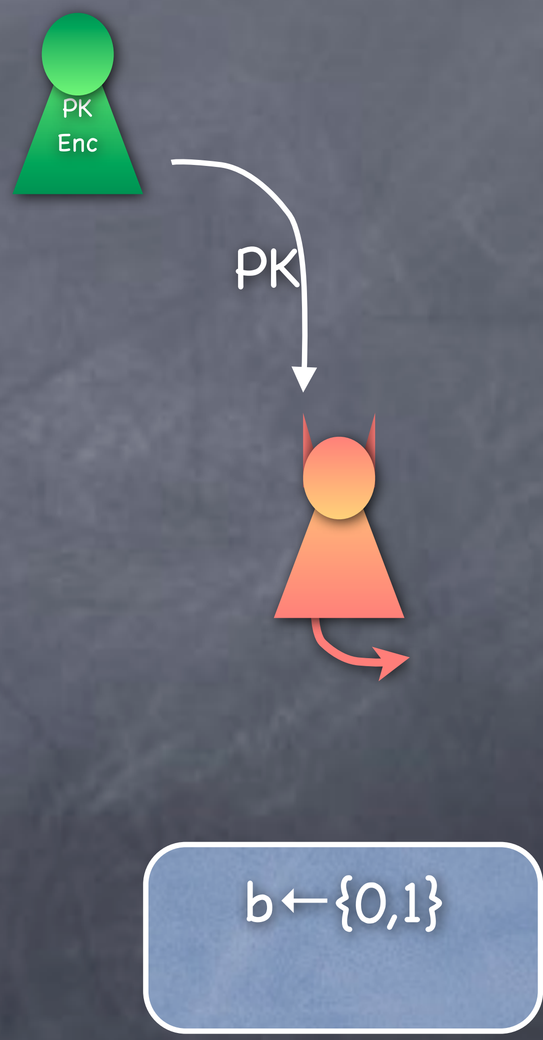

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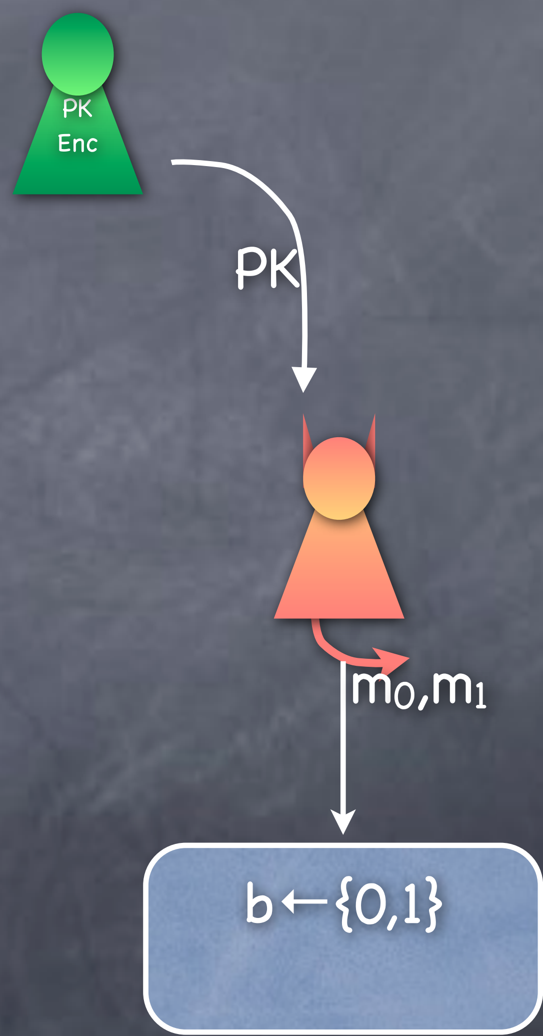

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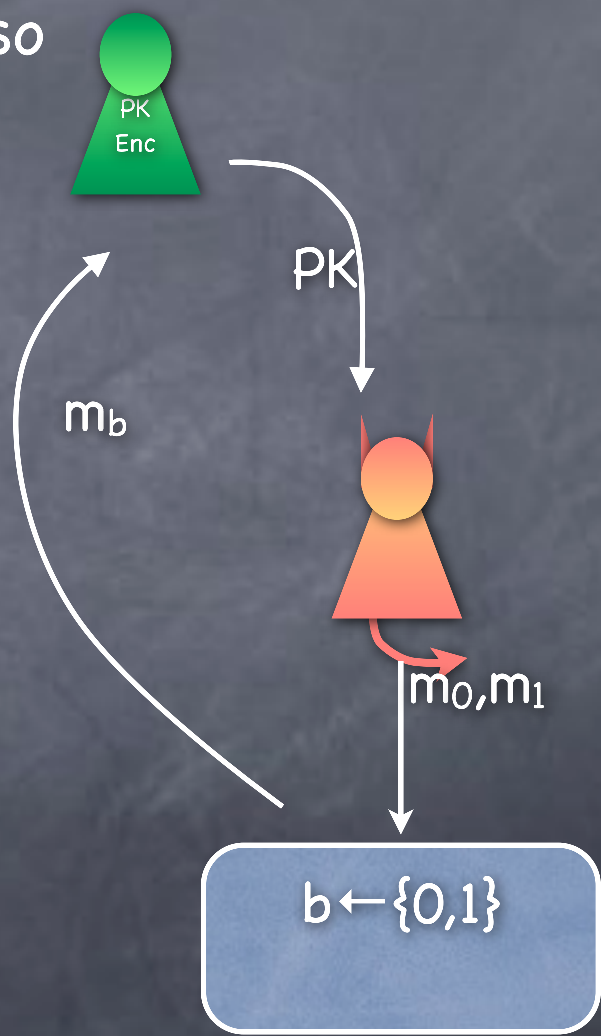

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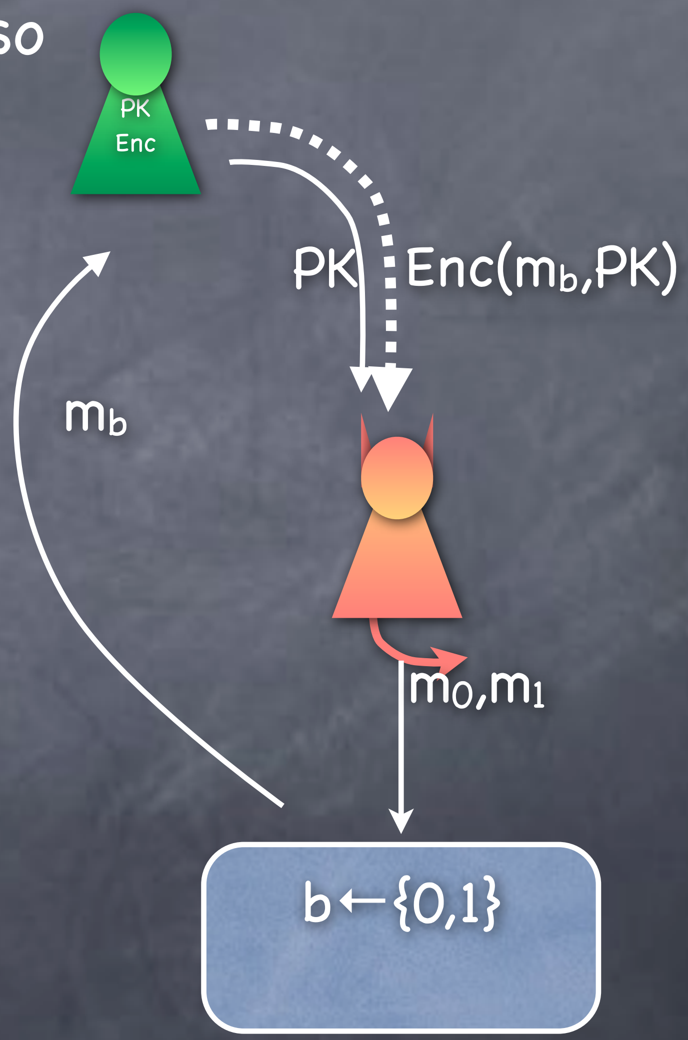

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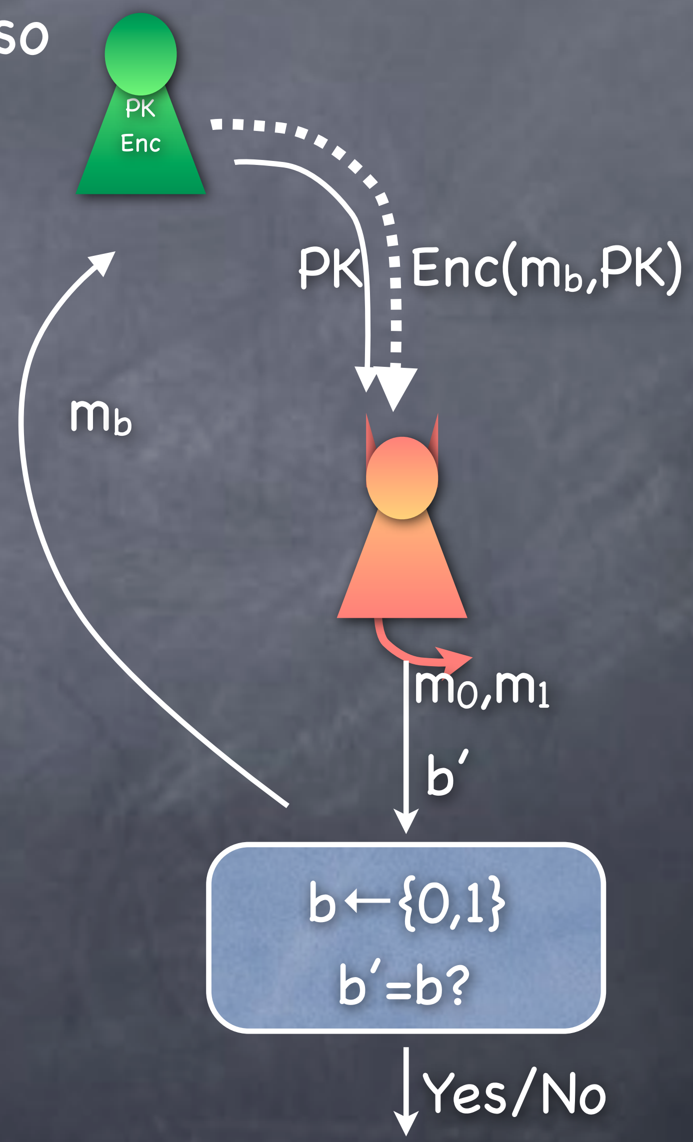

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Unless assumptions of imperfect eavesdropping

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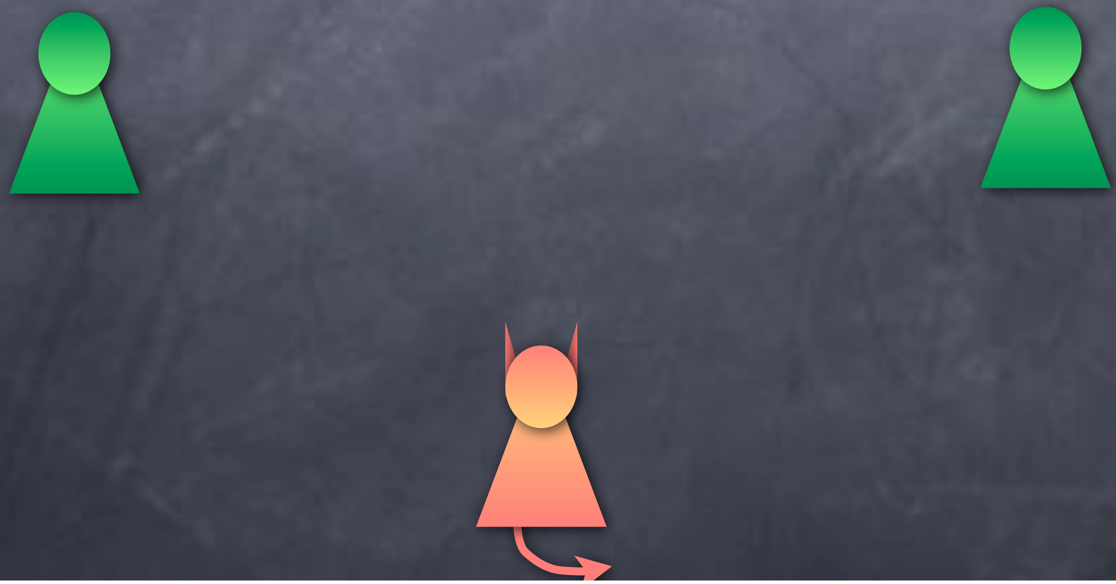

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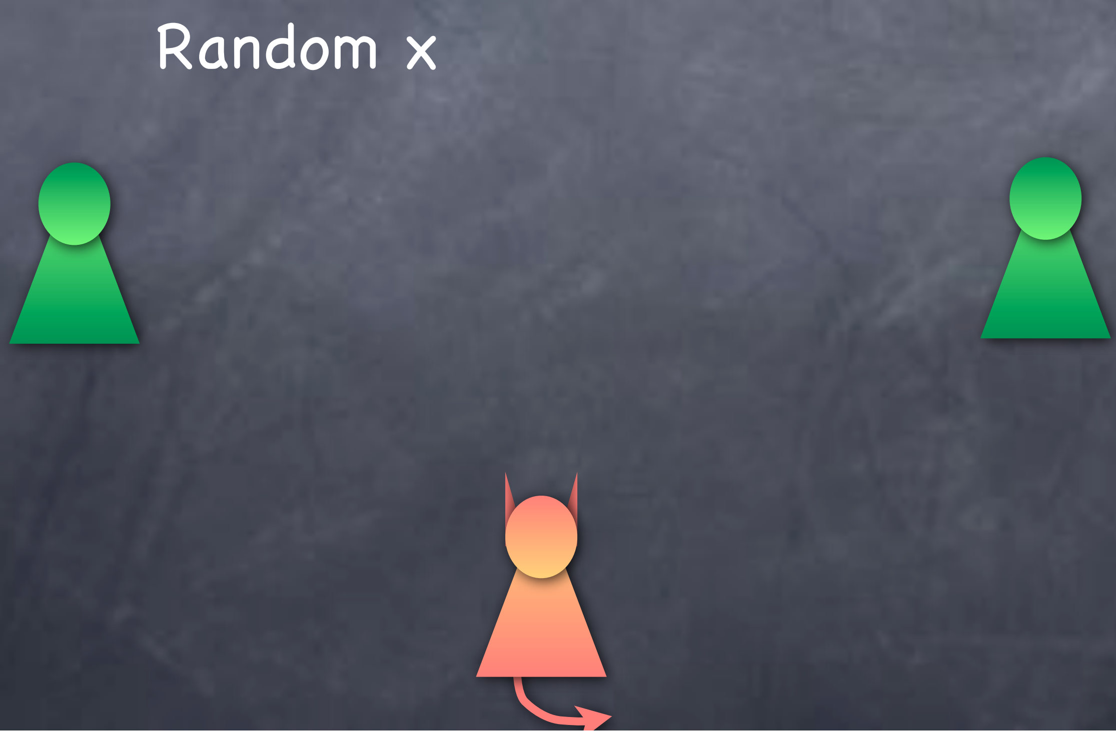

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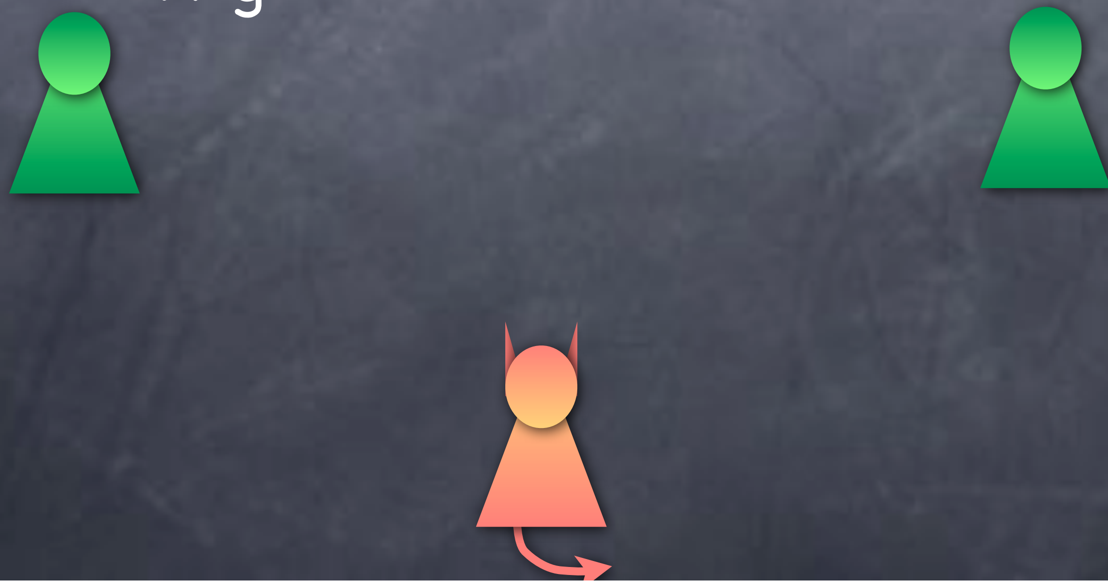
Random x

$X = g^x$



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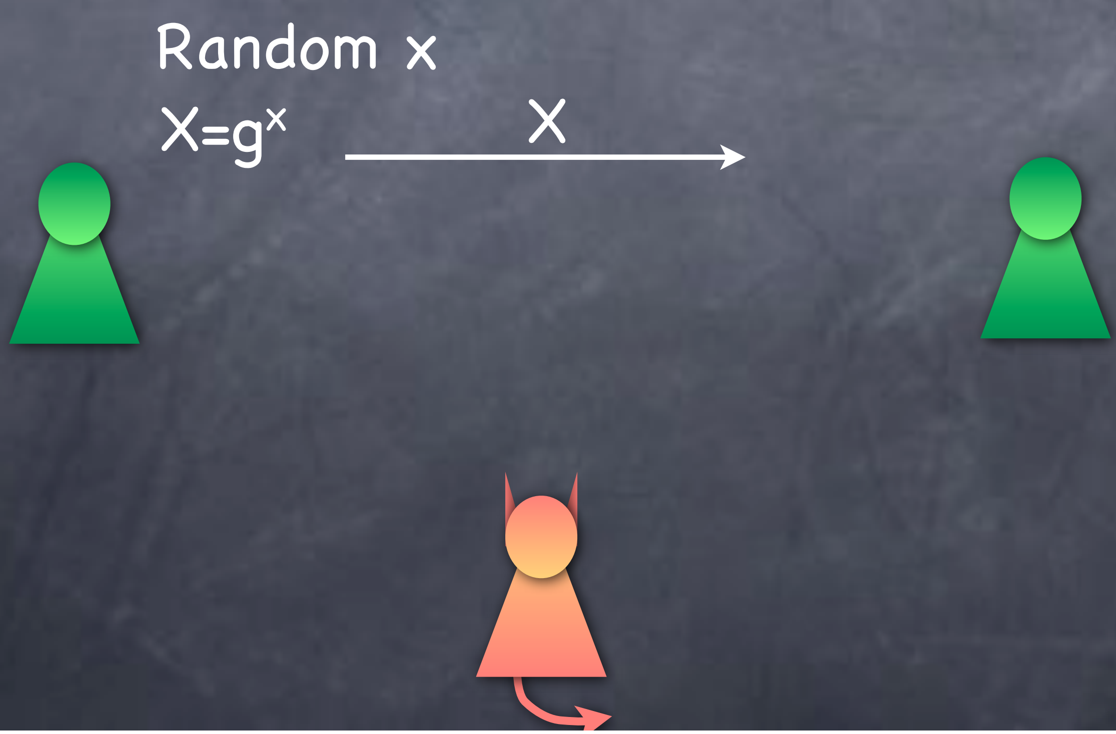

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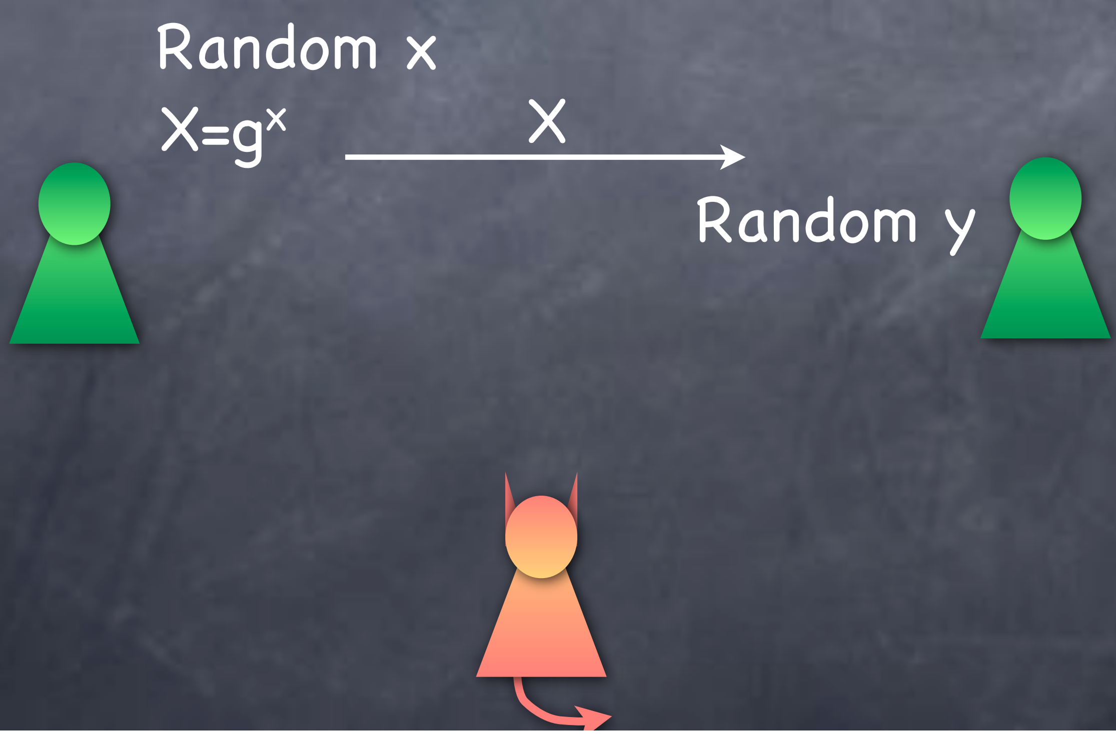

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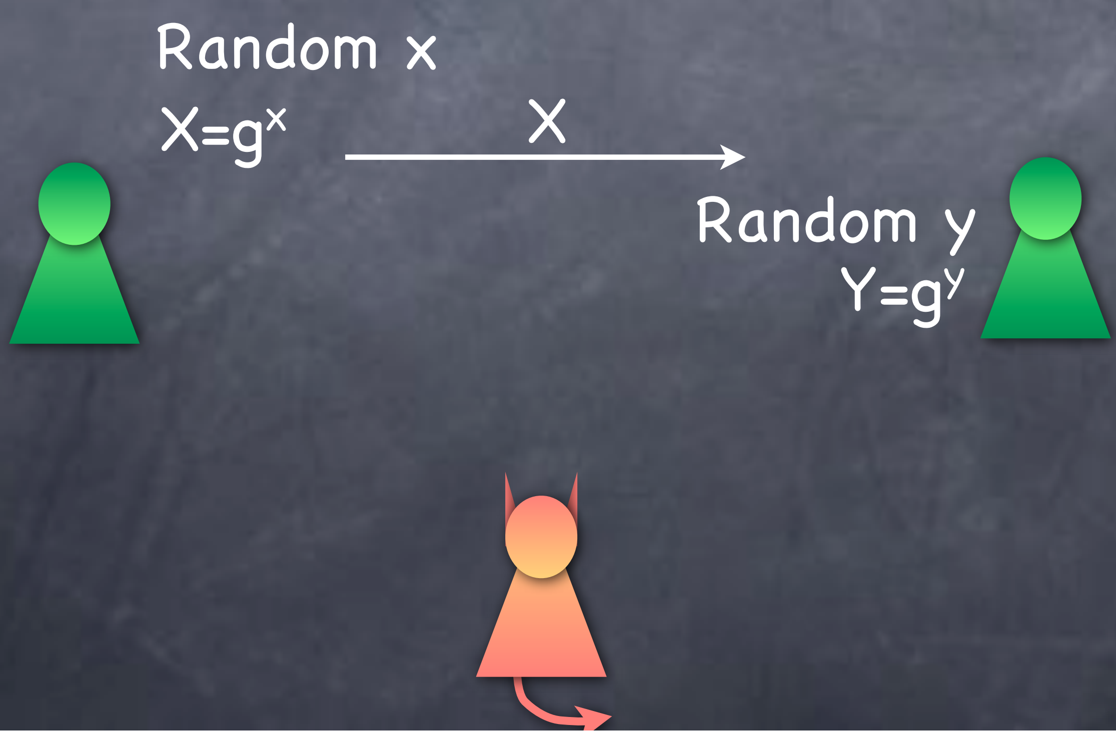

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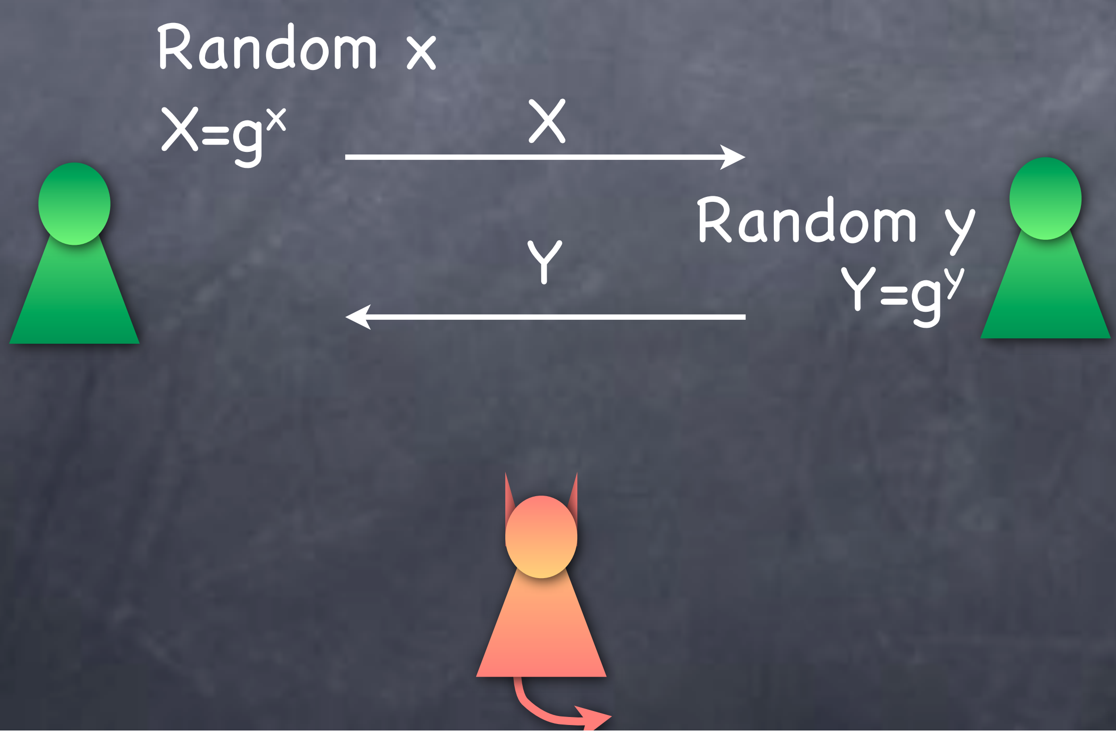

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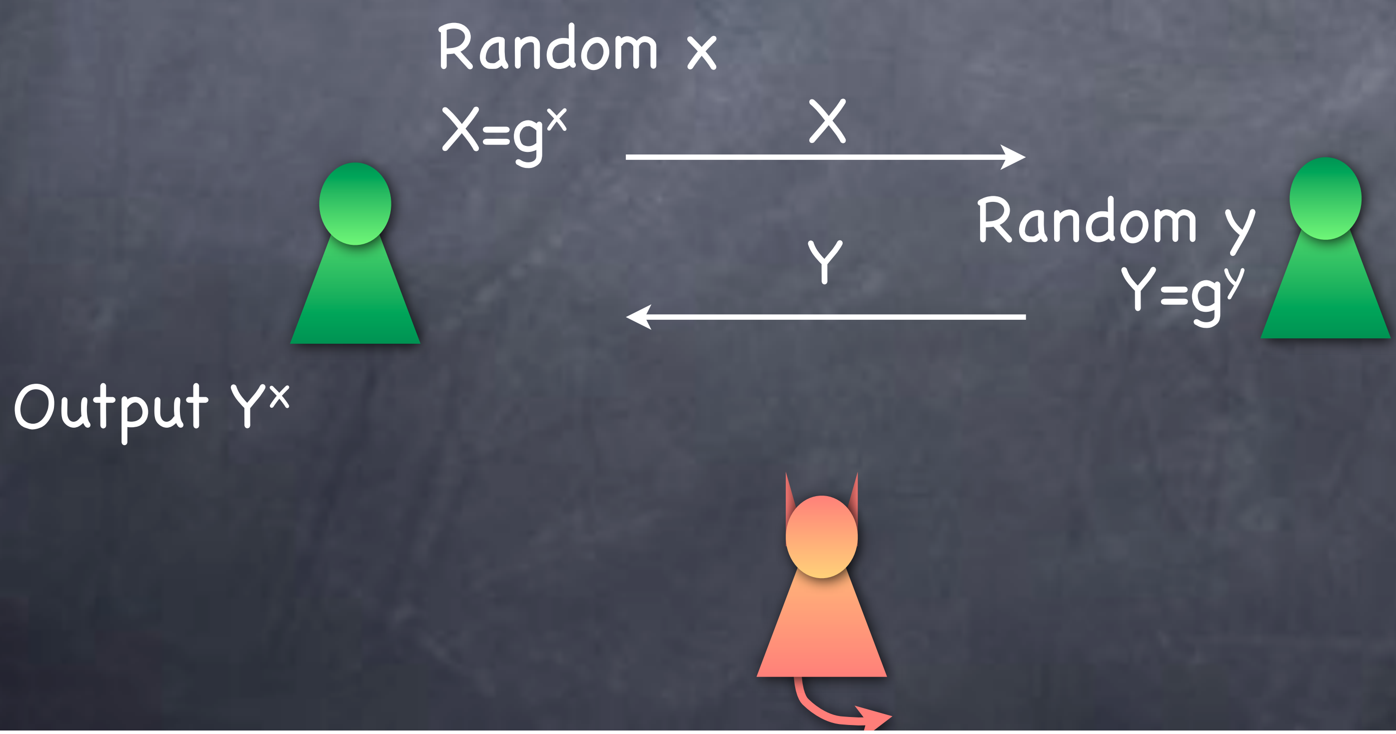

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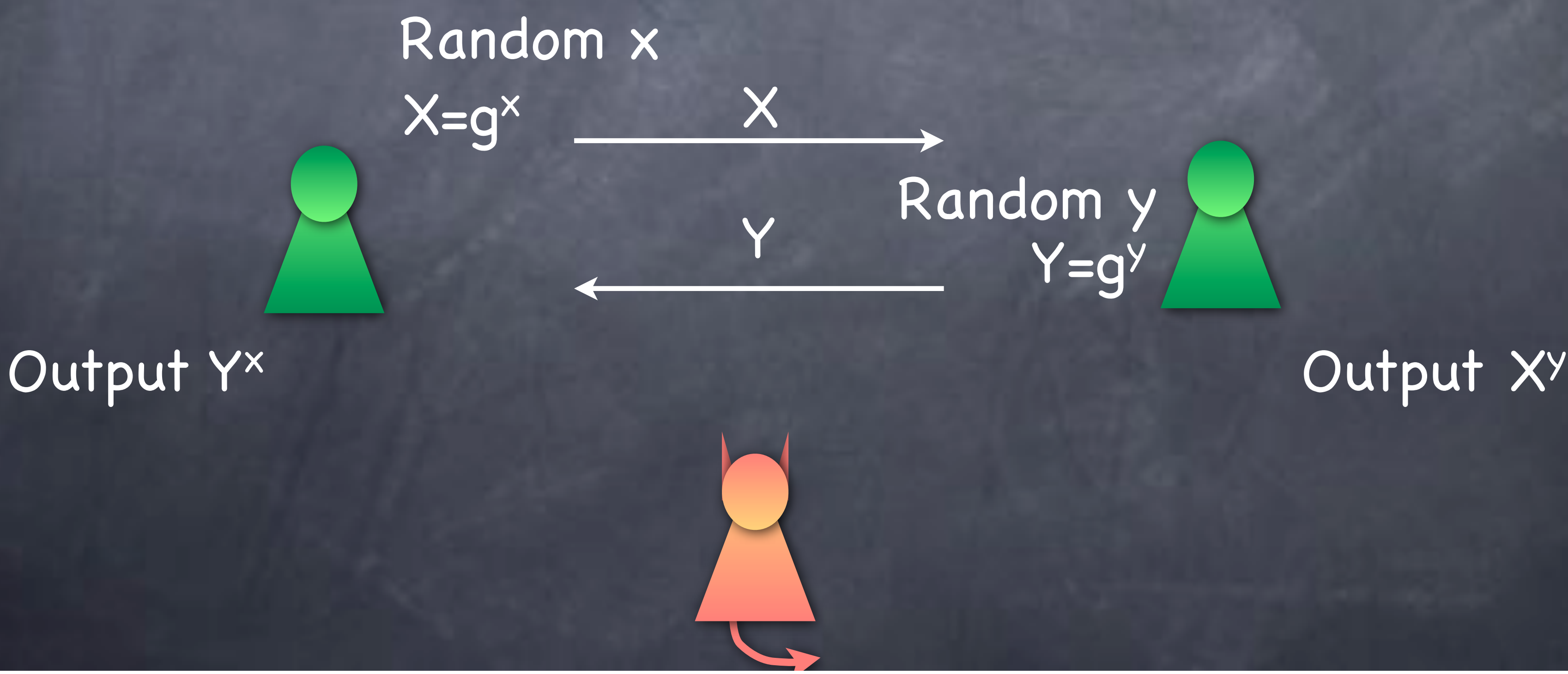

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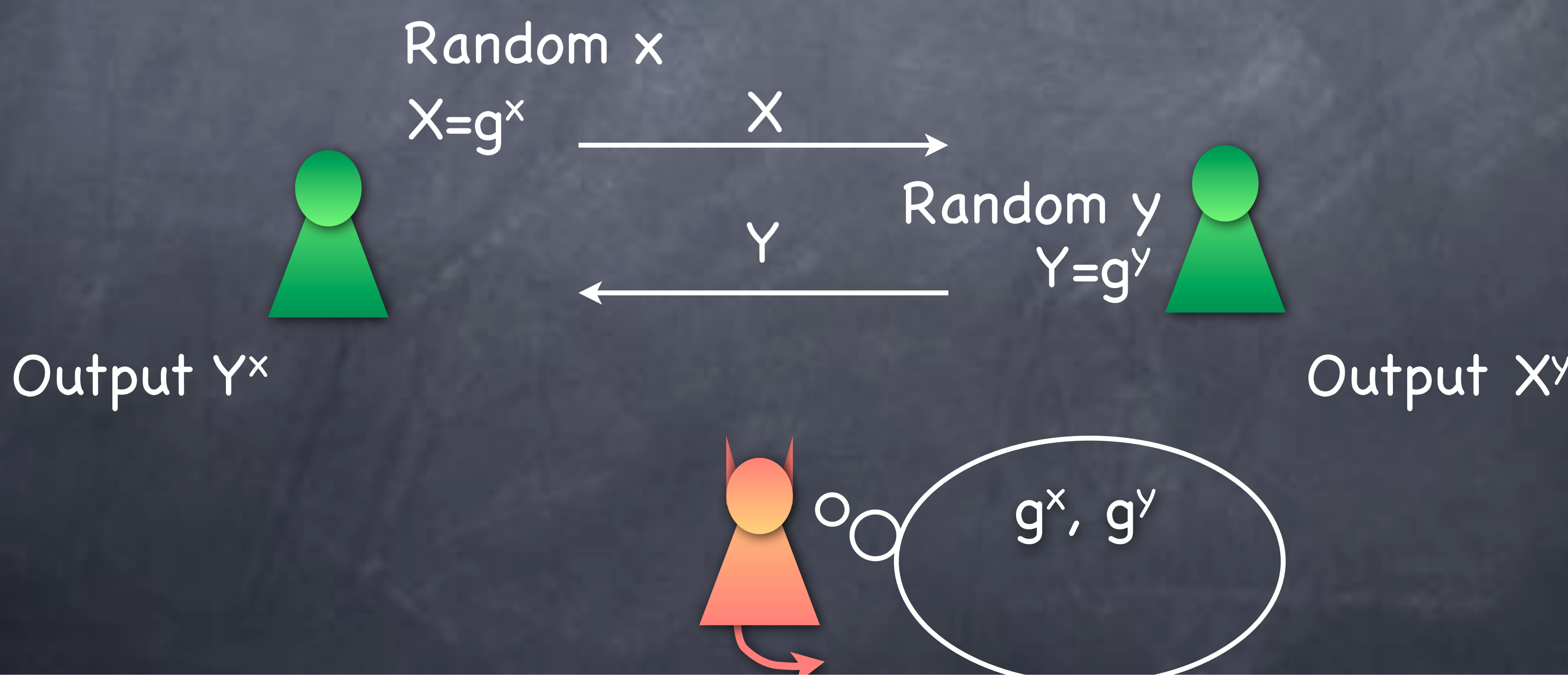

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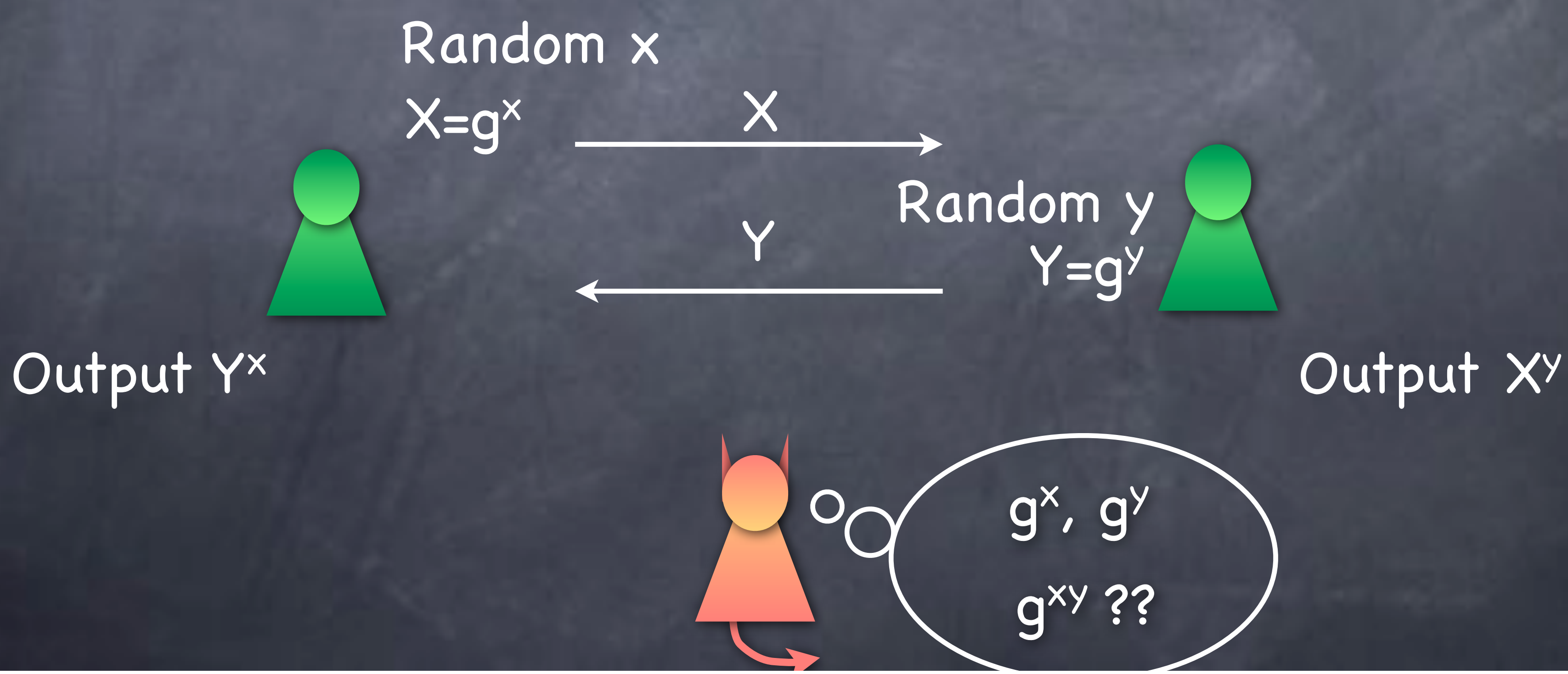

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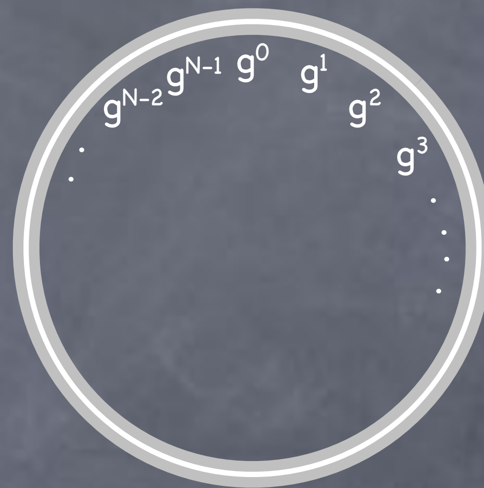

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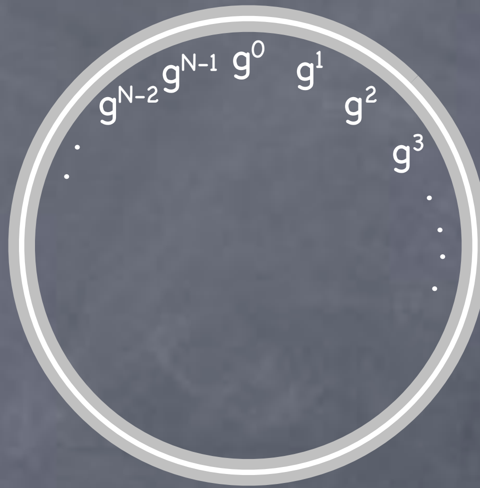

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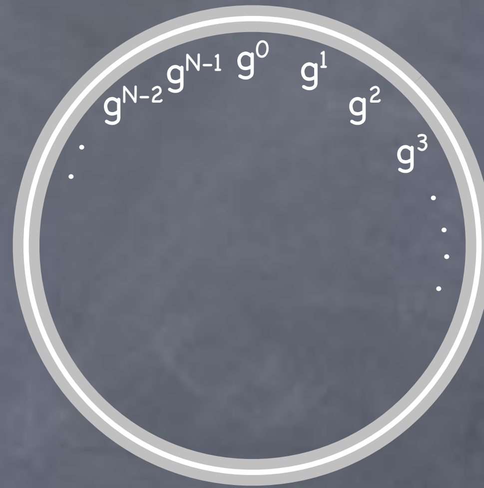

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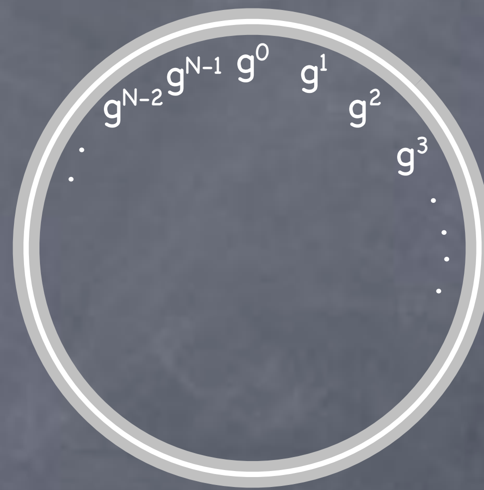

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    - (Also cyclic for certain other values of N)

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DDH Candidate:
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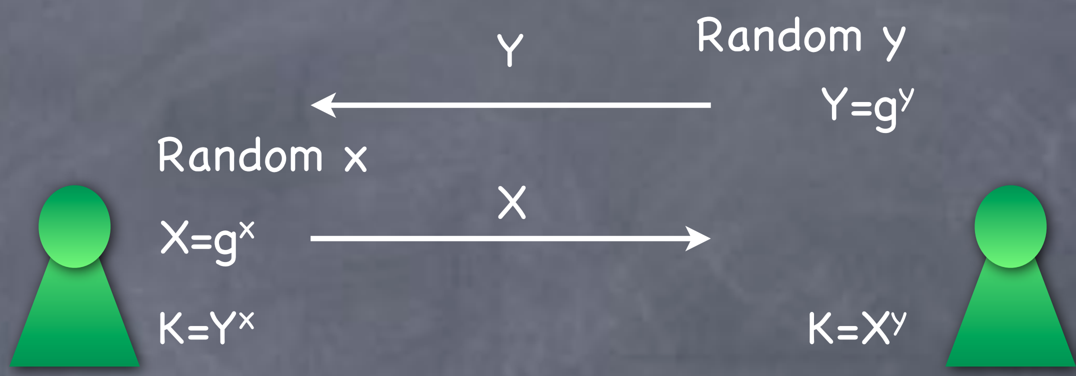

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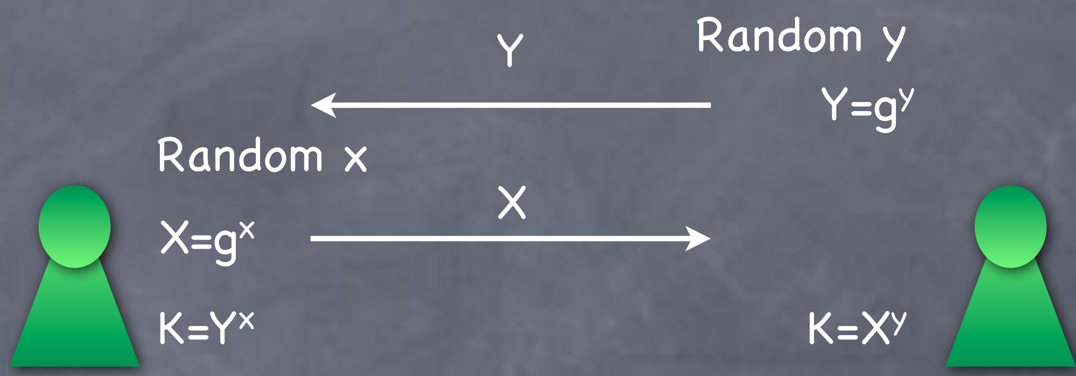

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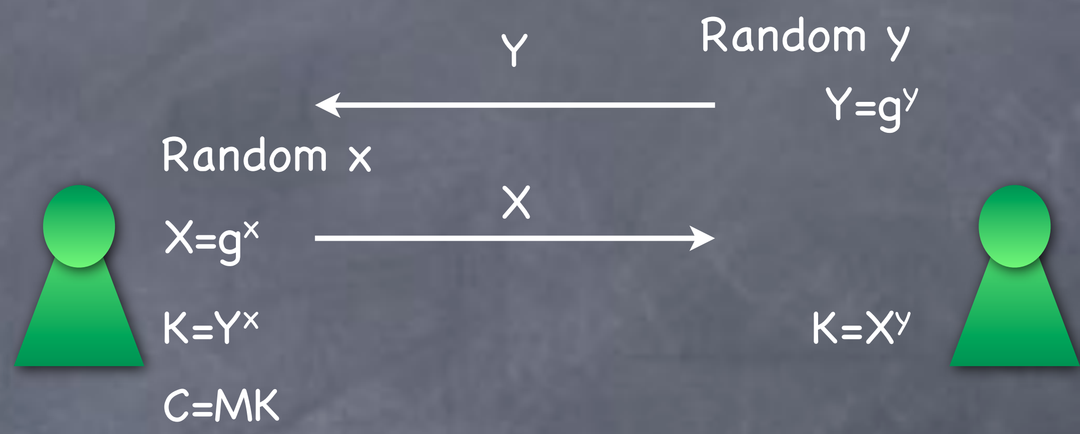

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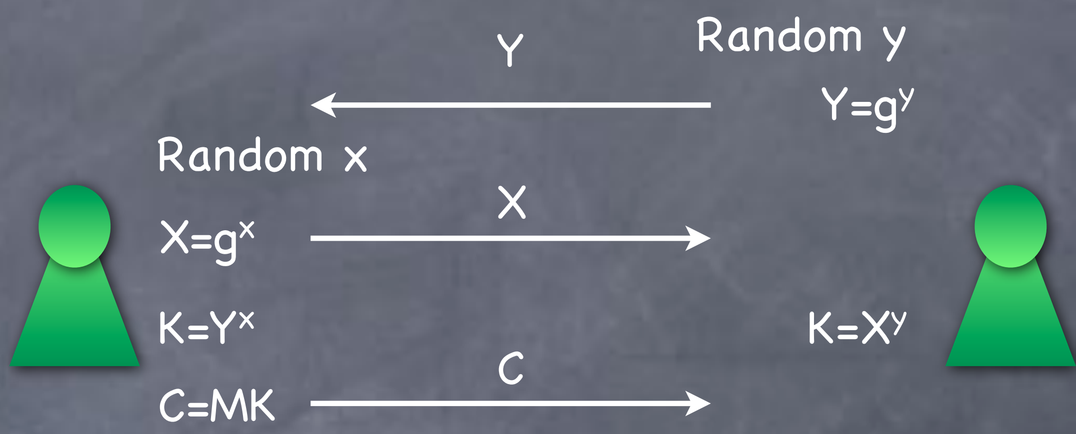

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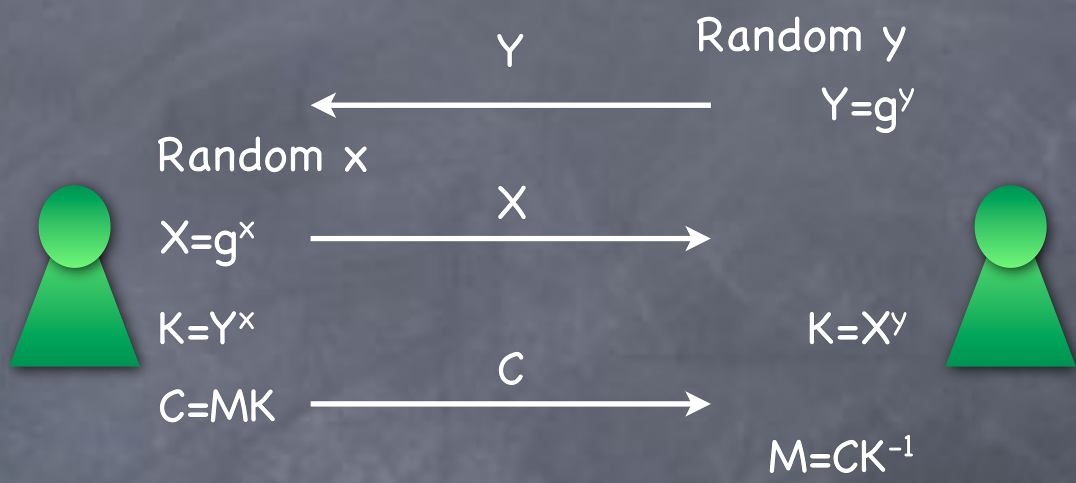

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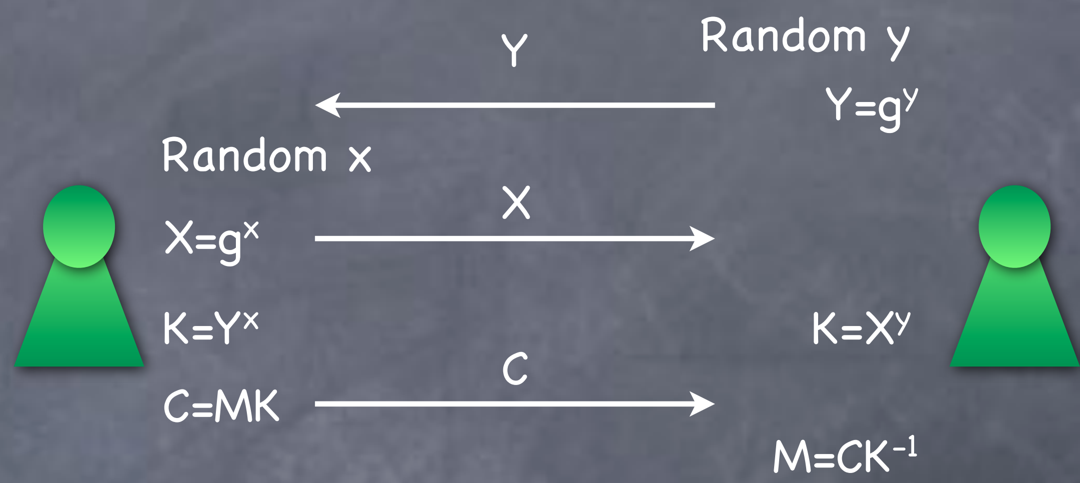

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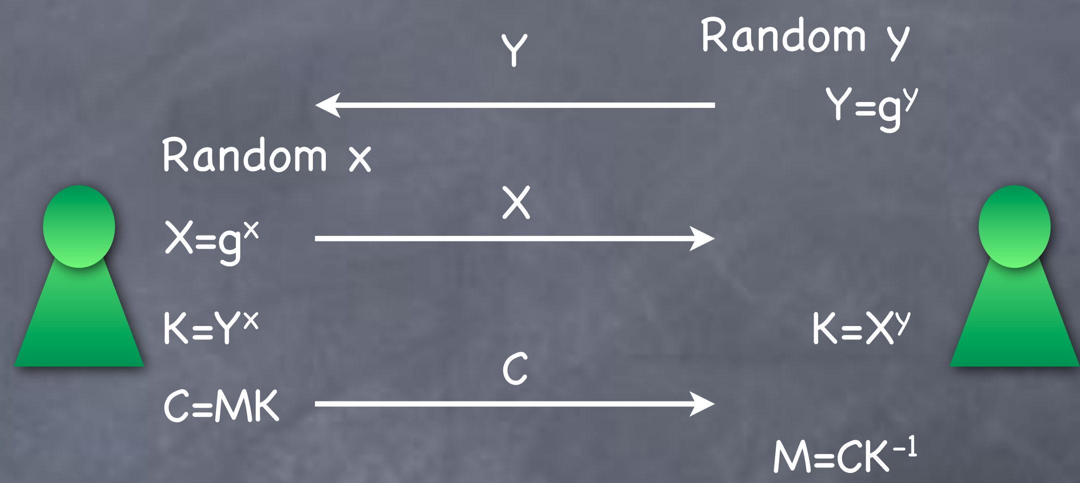

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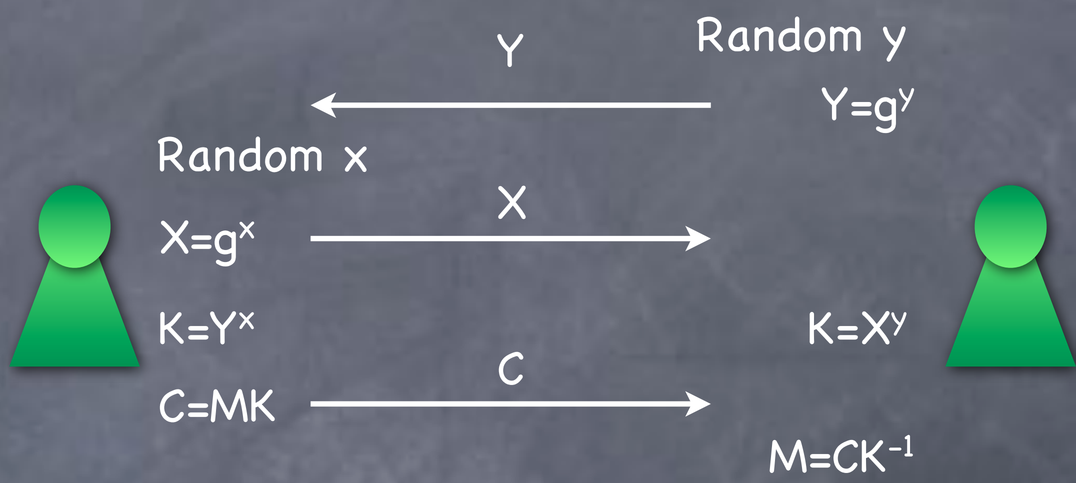

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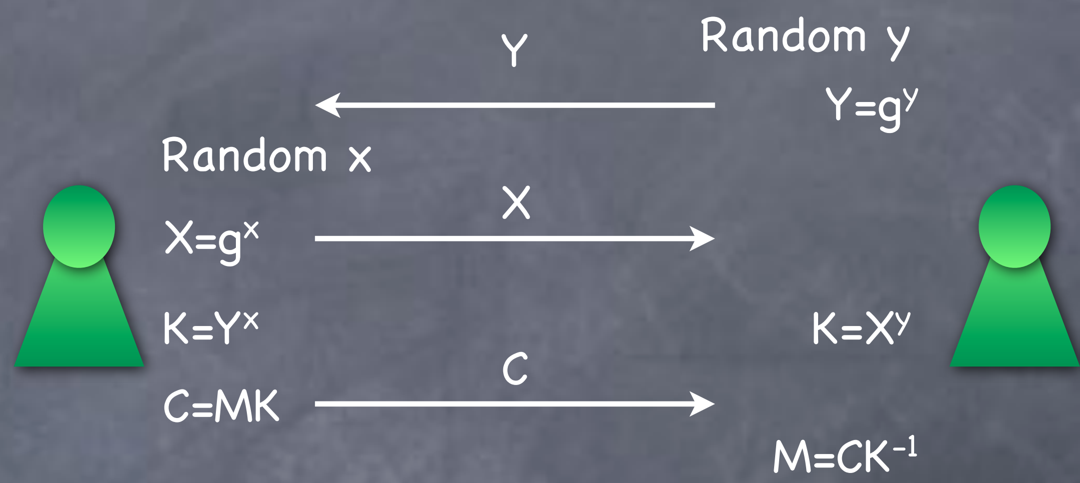

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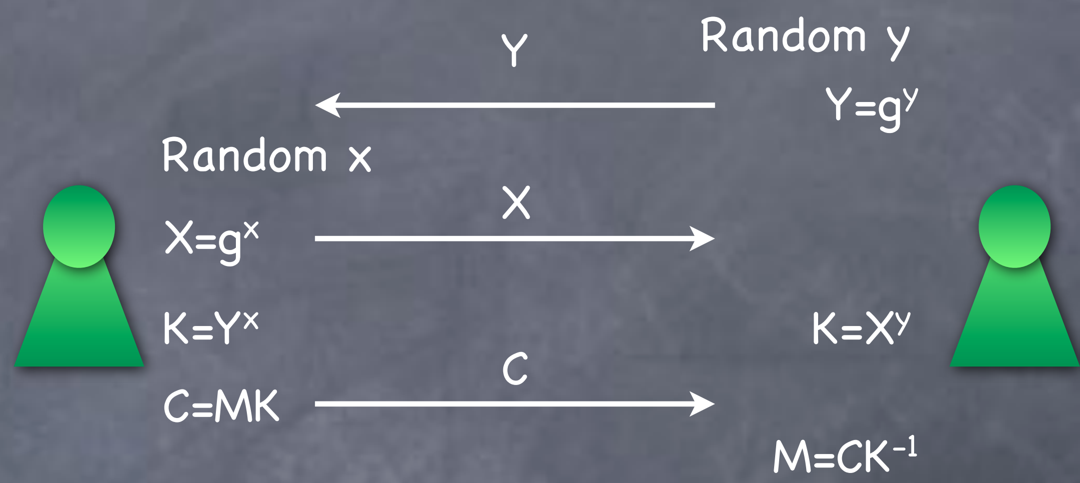

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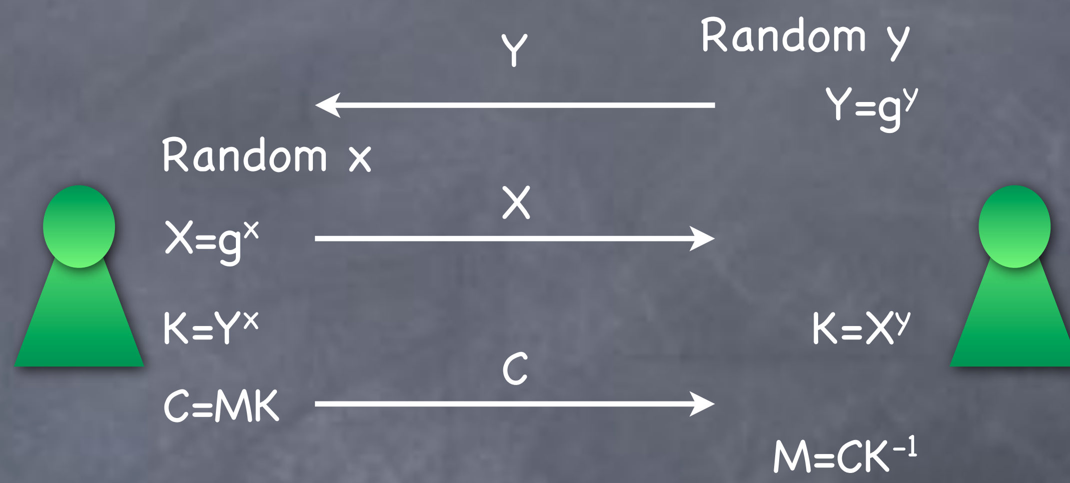

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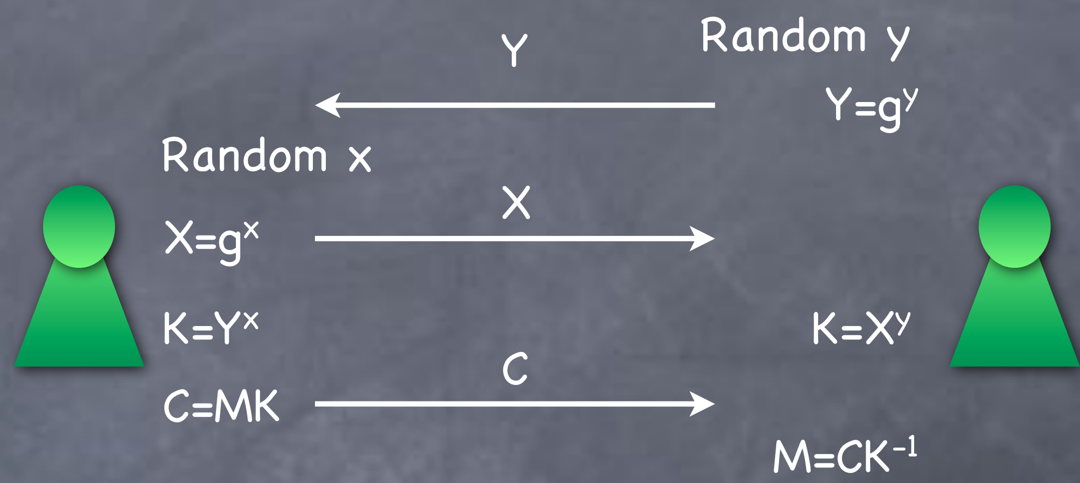

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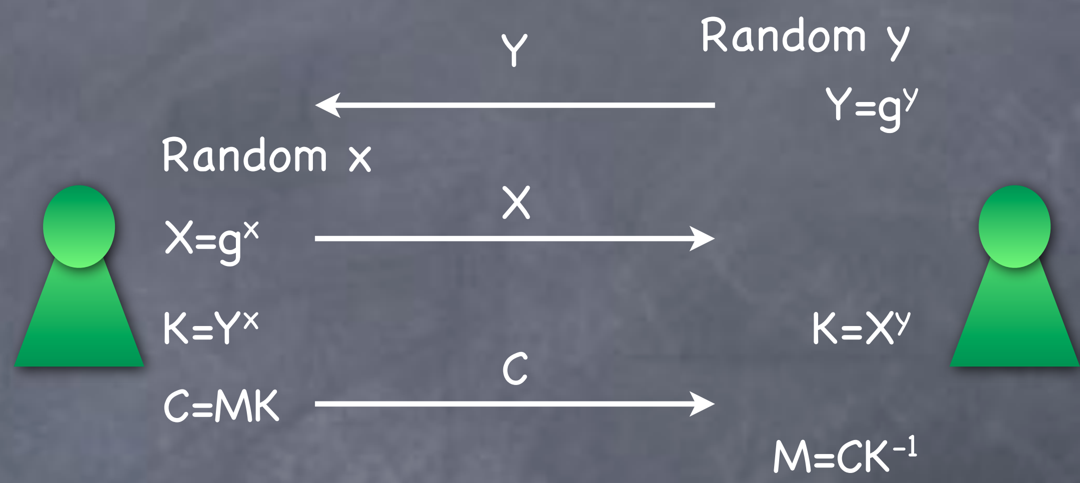

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  - When z=xy, exactly IND-CPA experiment: A* outputs 1 with probability = 1/2 + advantage of A.

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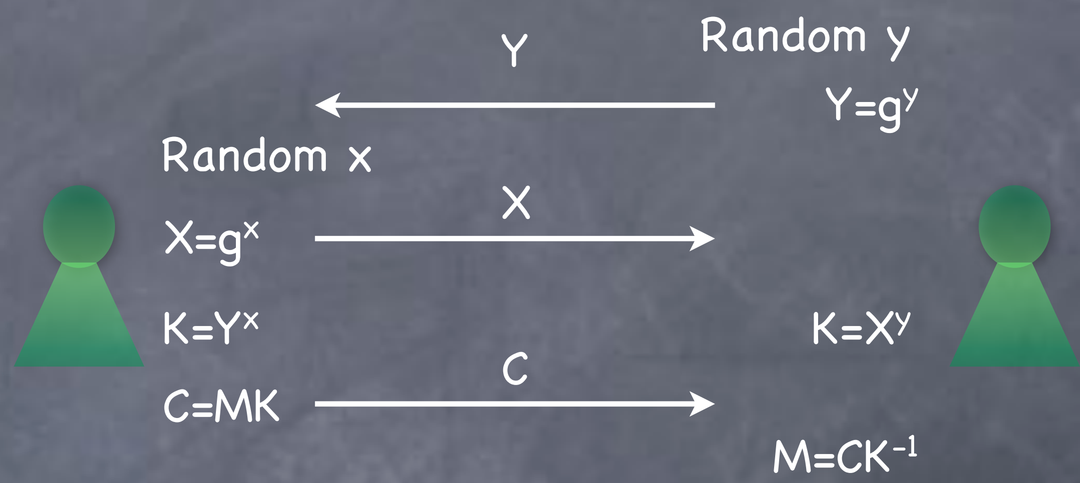

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Random y

$Y=g^y$

$\xleftarrow{Y}$

Random x

$X=g^x$ $\xrightarrow{X}$

$K=Y^x$ $\qquad$ $K=X^y$

$C=MK$ $\xrightarrow{C}$

$M=CK^{-1}$

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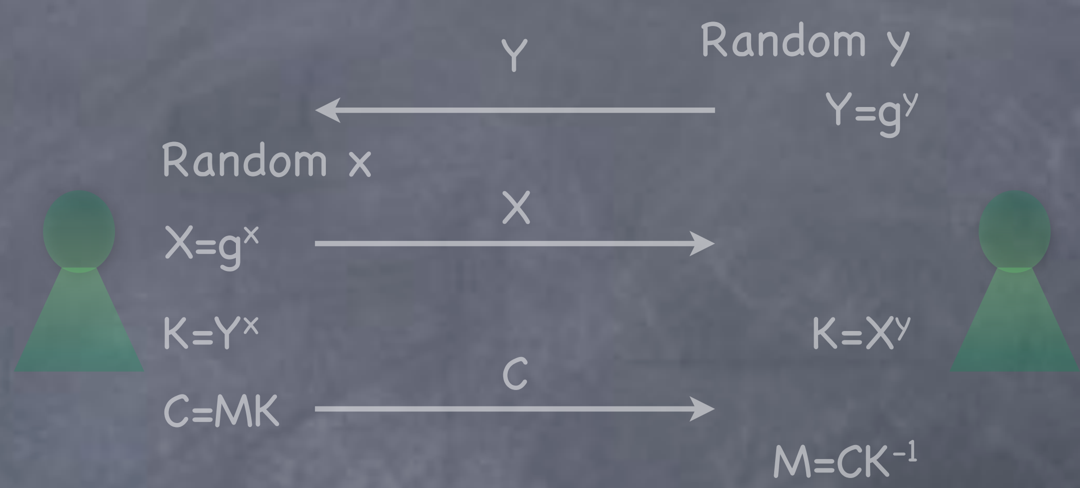

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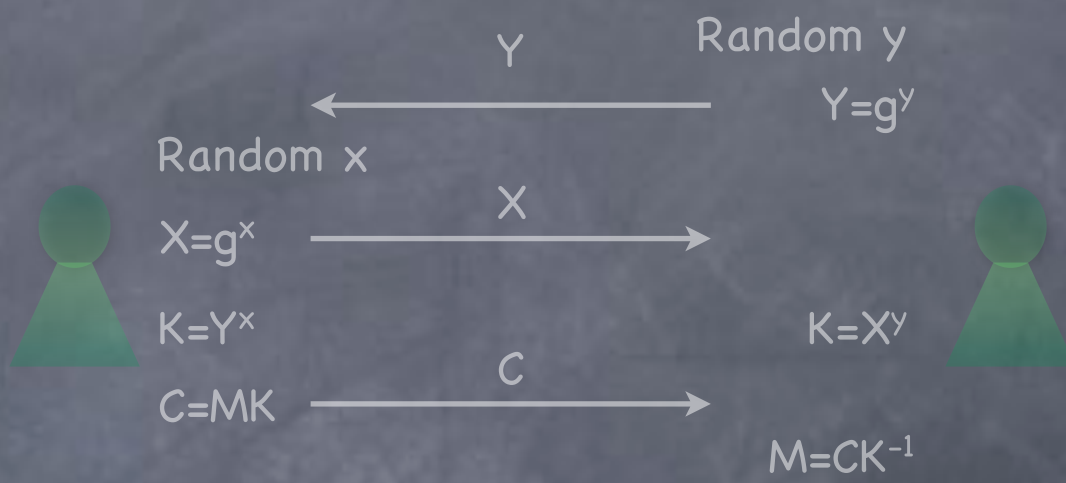

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Abstracting El Gamal

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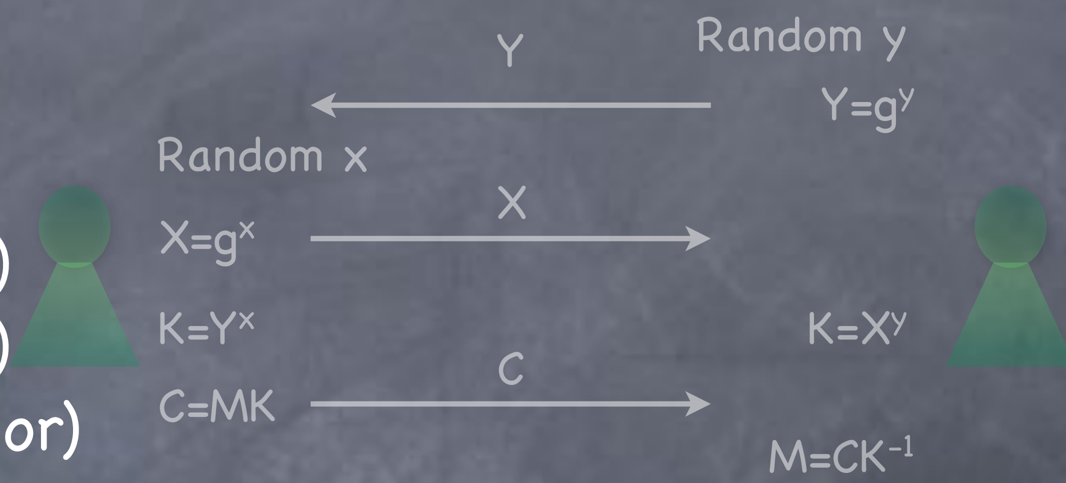

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Random y

$Y=g^y$

$\xleftarrow{\quad Y \quad}$

Random x

$X=g^x$ $\xrightarrow{\quad X \quad}$

$K=Y^x$ $\qquad$ $K=X^y$

$C=MK$ $\xrightarrow{\quad C \quad}$

$M=CK^{-1}$

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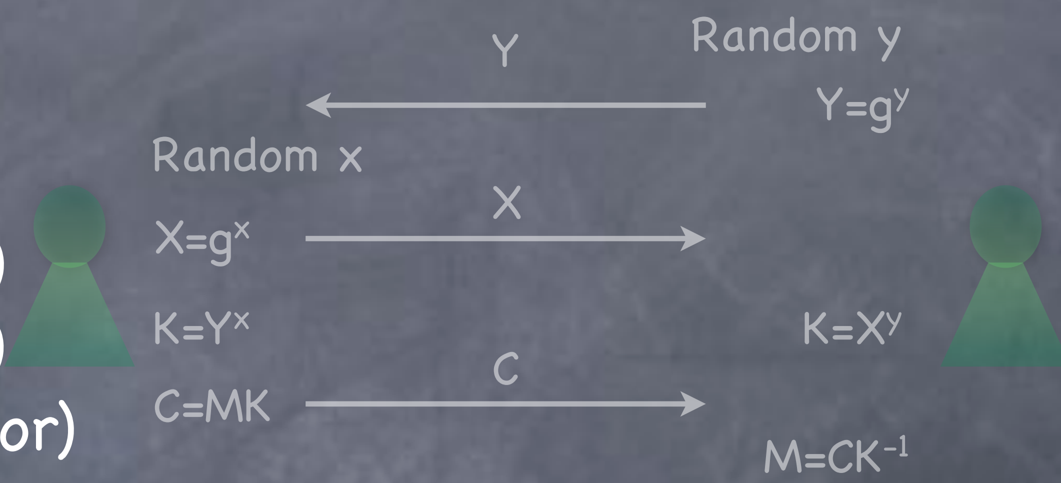

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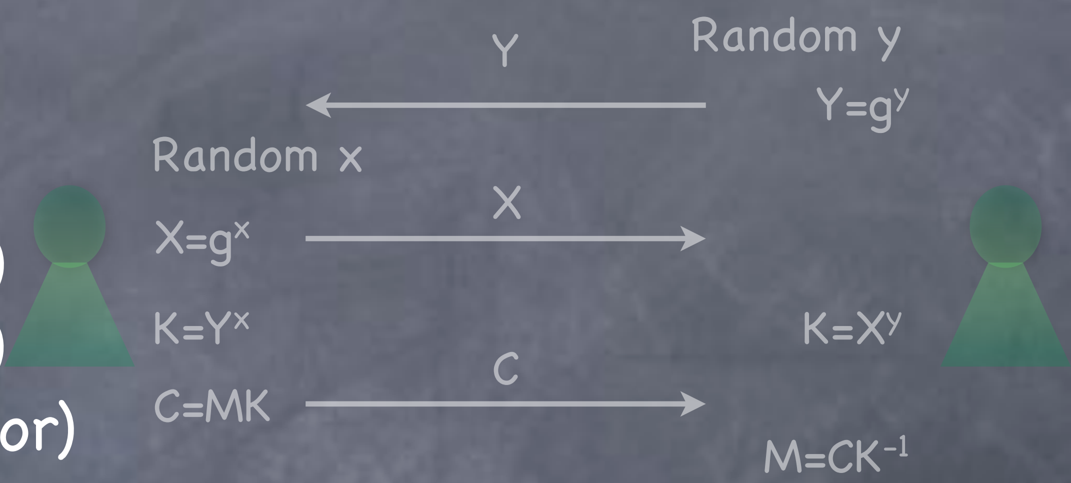

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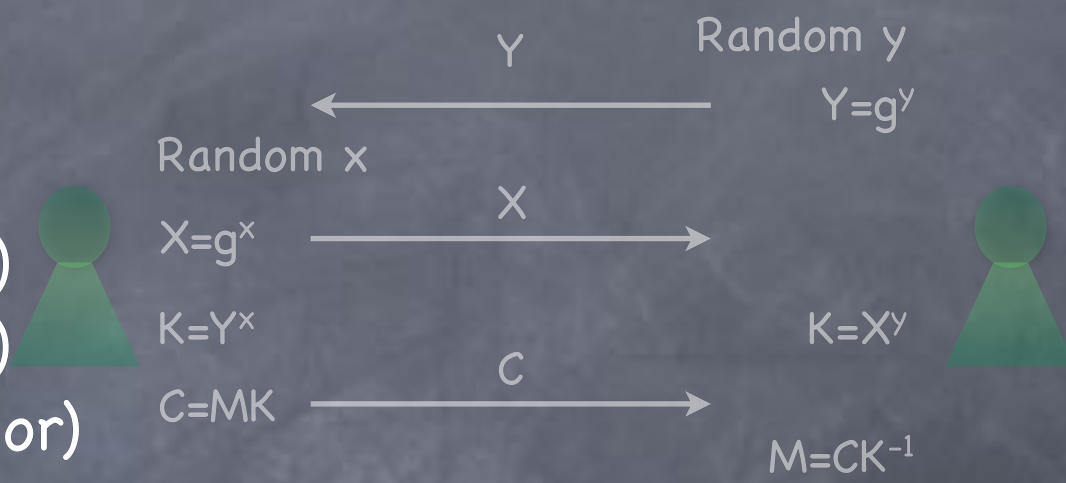

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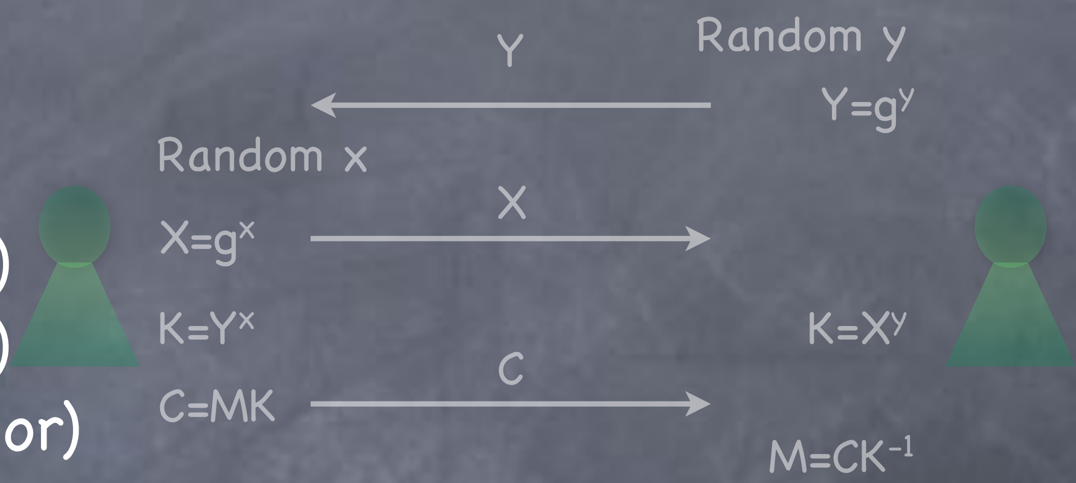

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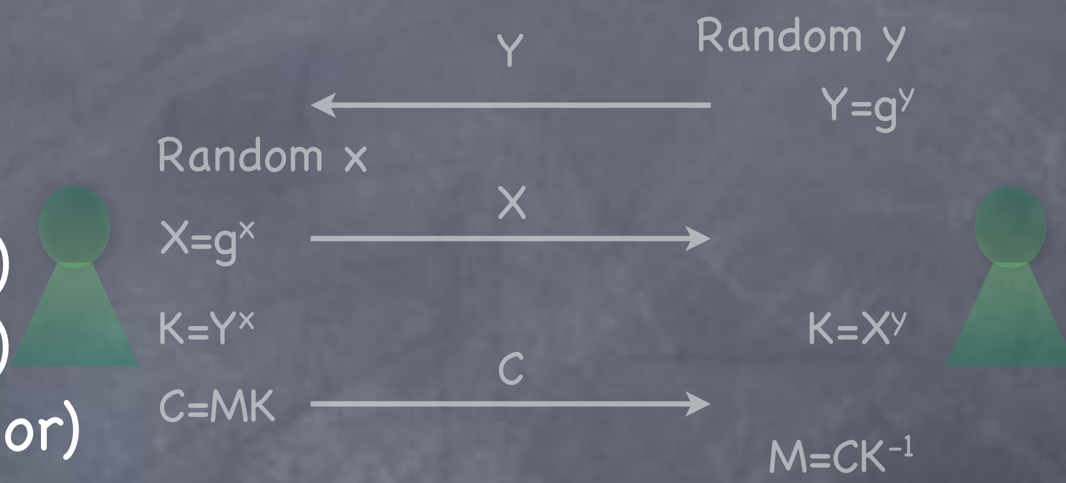

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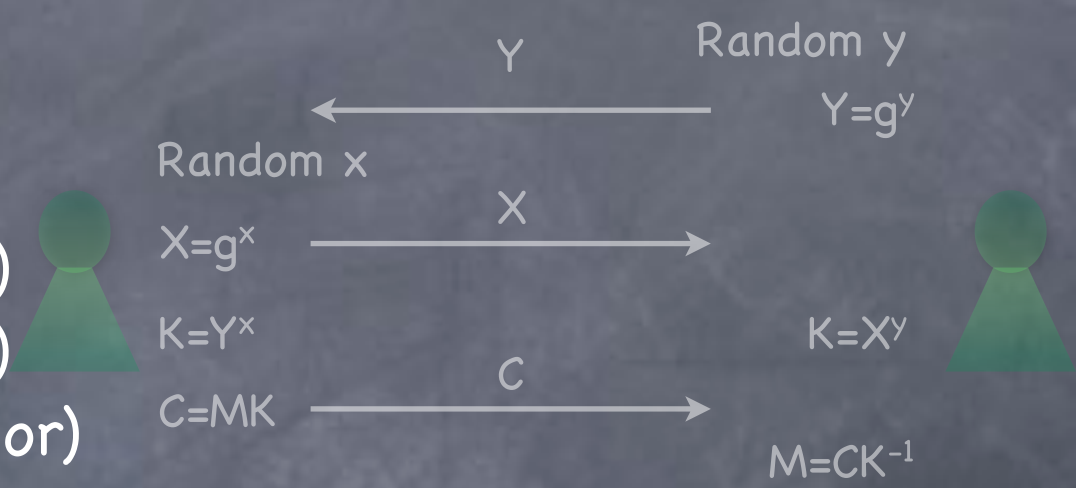

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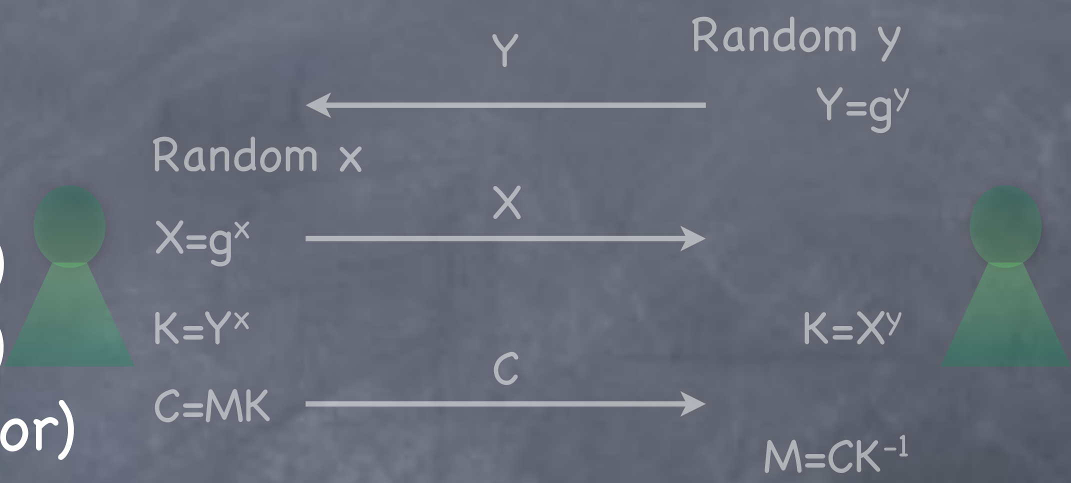
- Enough for an IND-CPA secure PKE scheme (cf. Security of El Gamal)



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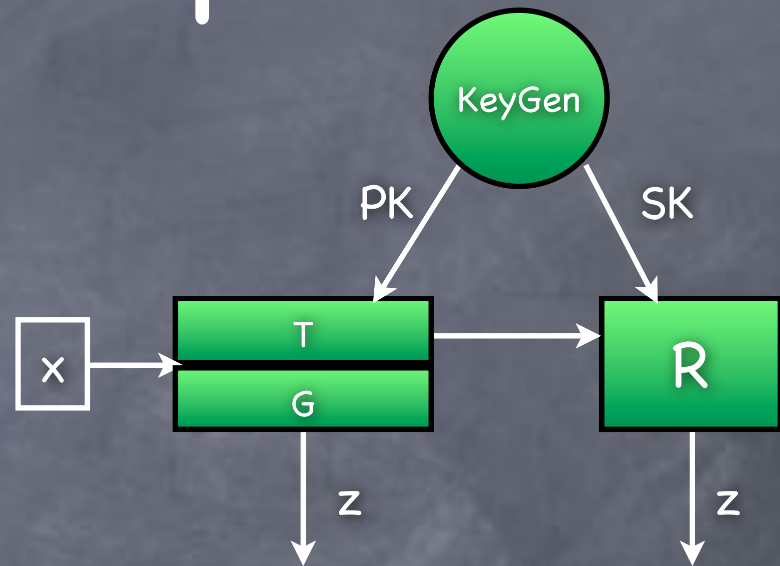
Trapdoor PRG from Generic Assumption?



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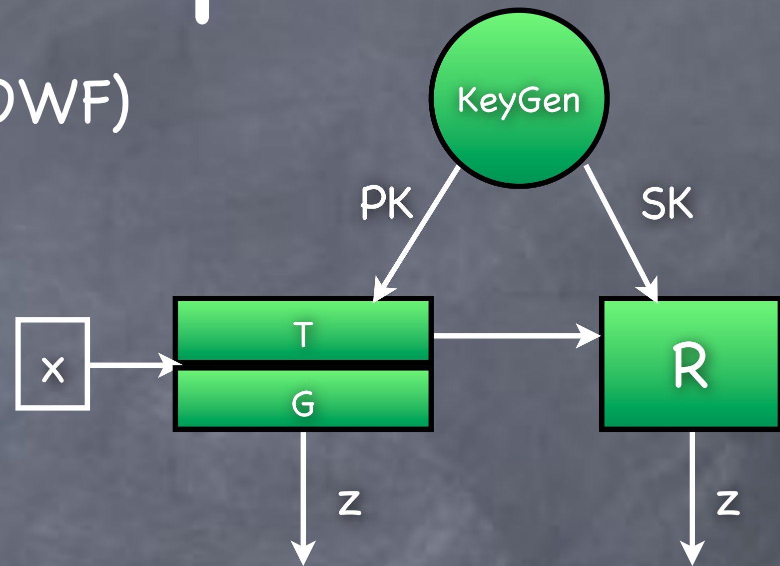

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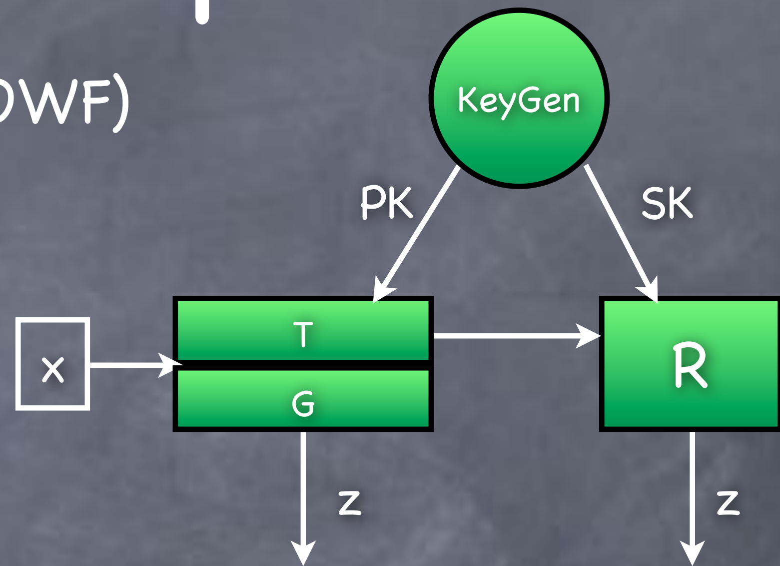

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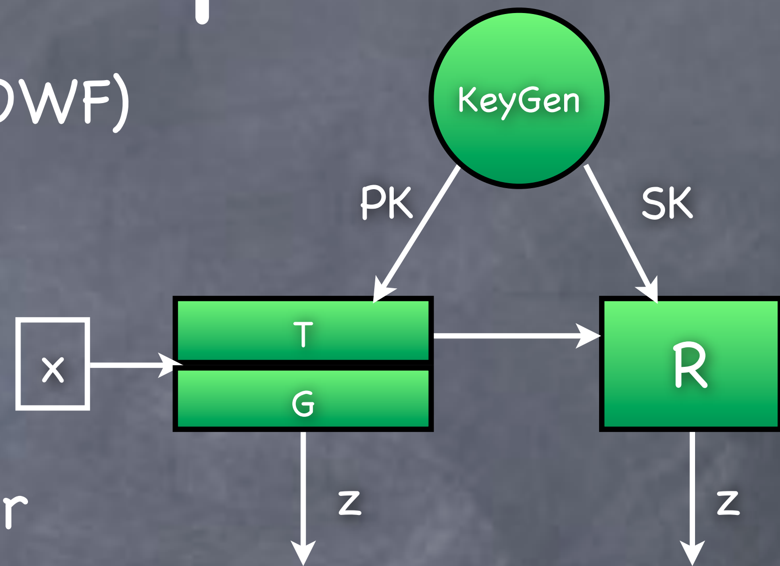

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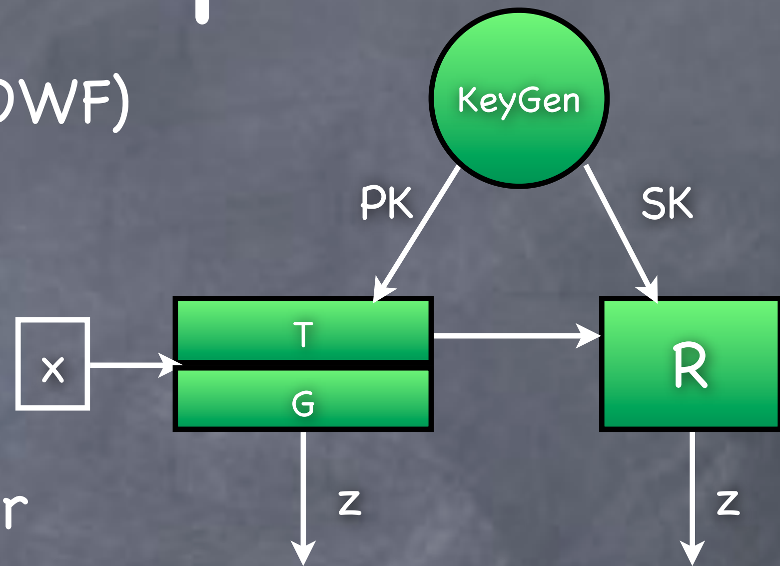

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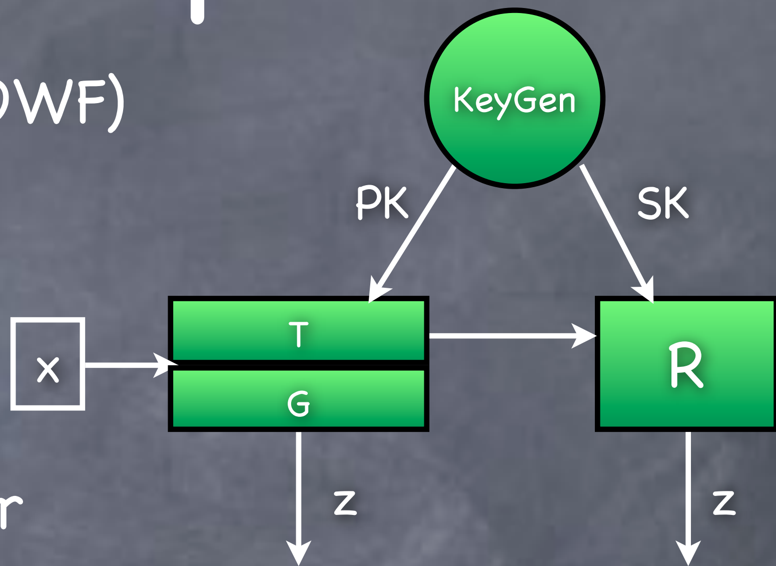
  - Will start with "Trapdoor OWP"



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Trapdoor OWP

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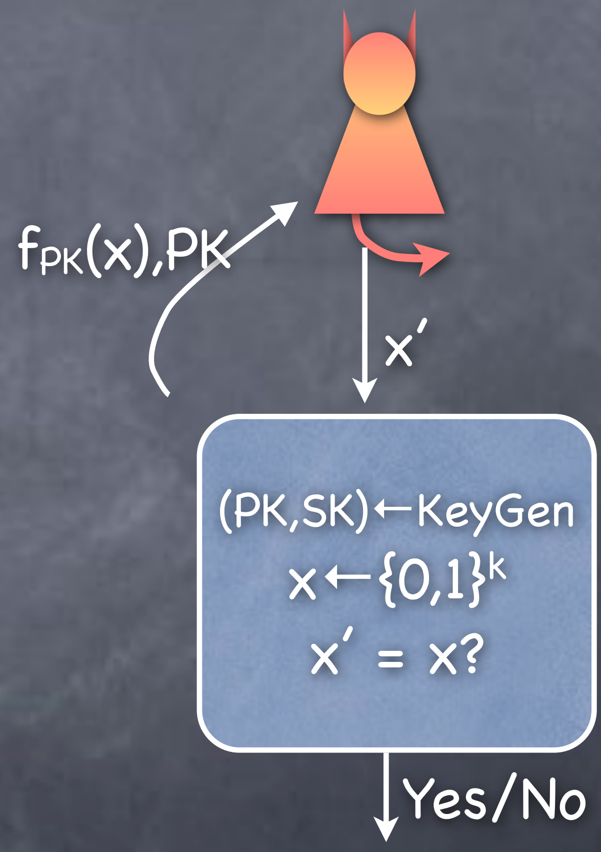

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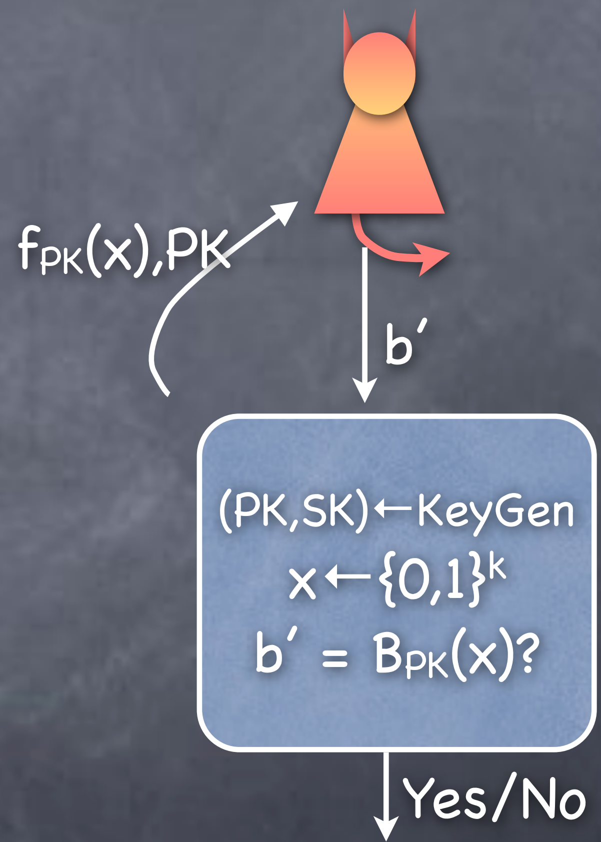
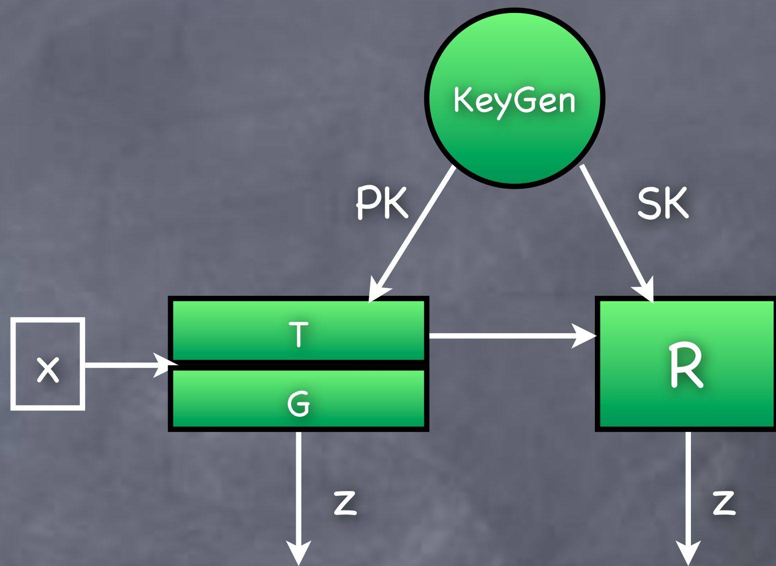
  - Hardcore predicate:
    - B_{PK} s.t. $(PK,f_{PK}(x),B_{PK}(x)) \approx (PK,f_{PK}(x),r)$



Trapdoor PRG from Trapdoor OWP



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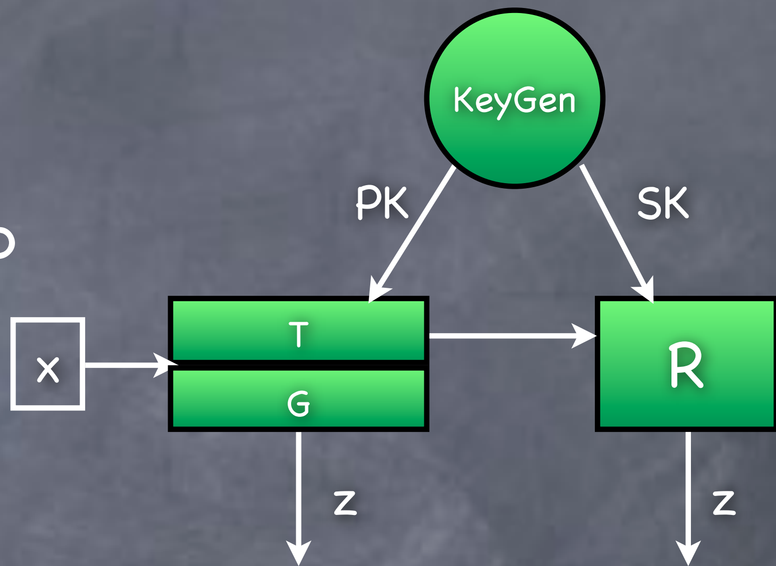

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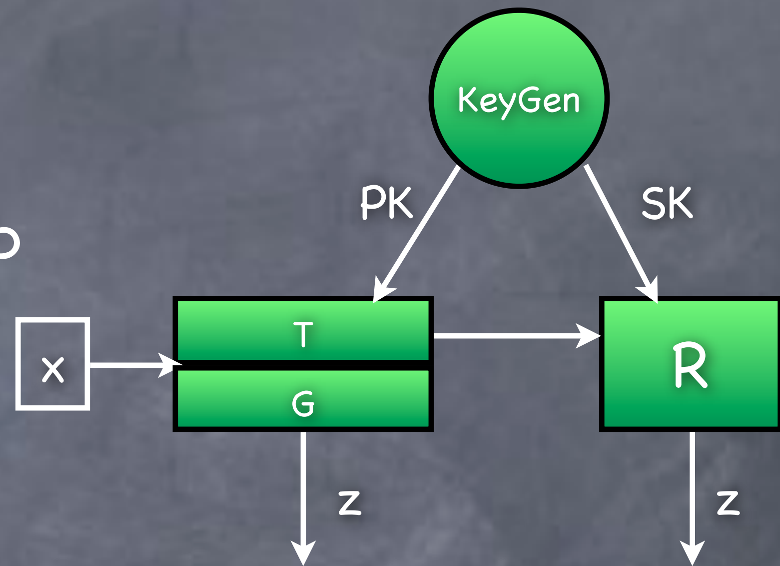

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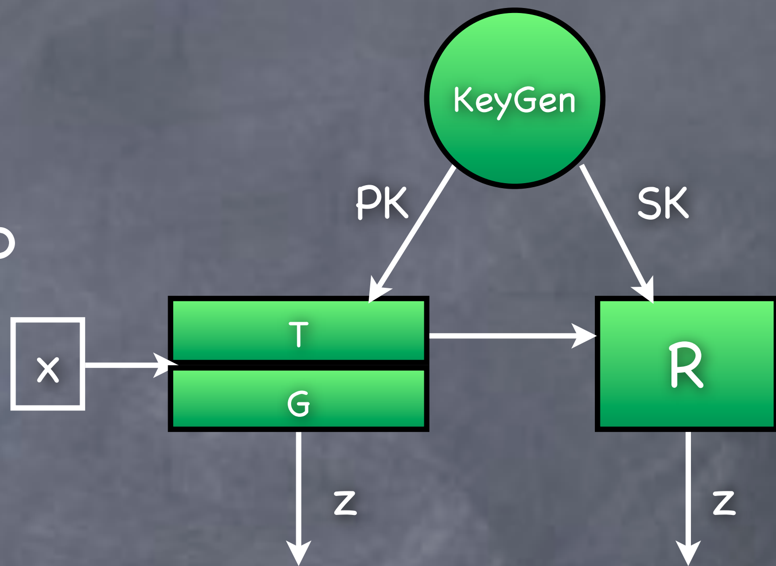

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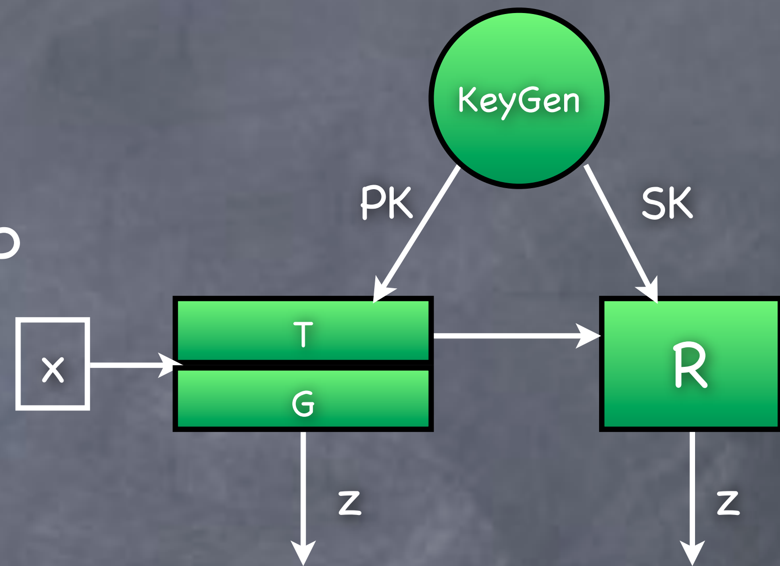

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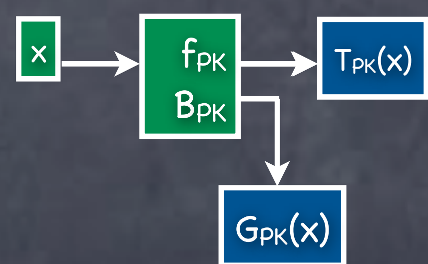

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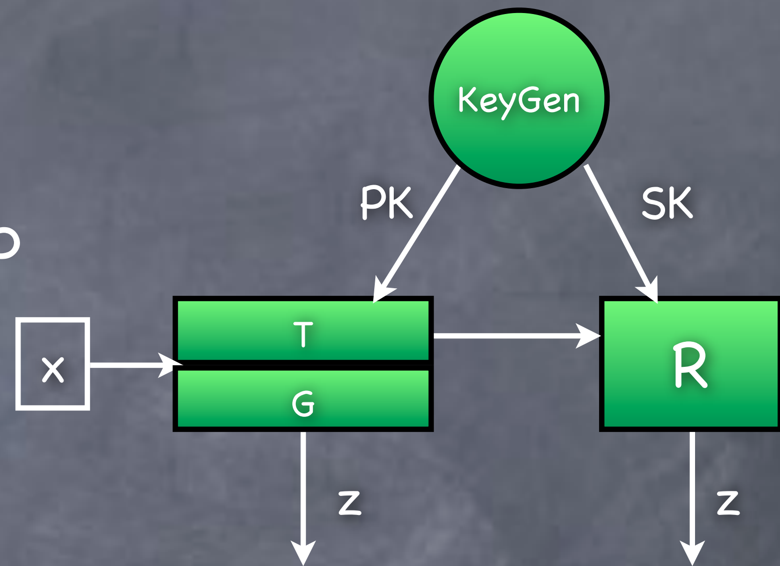

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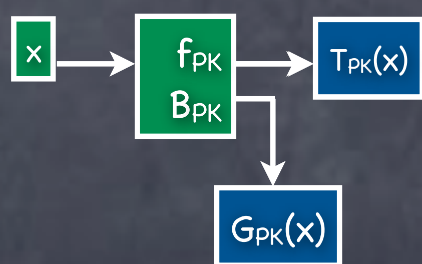

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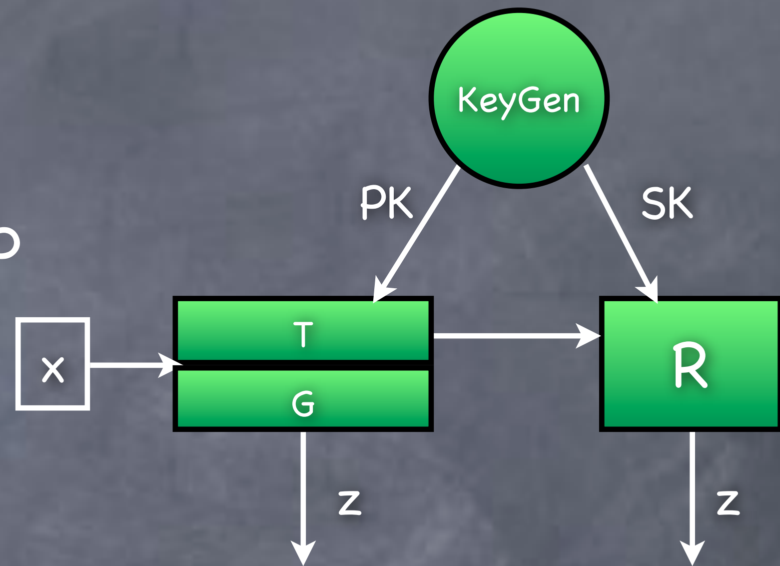

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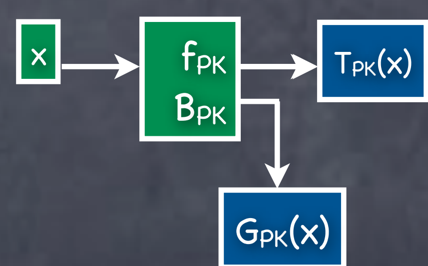

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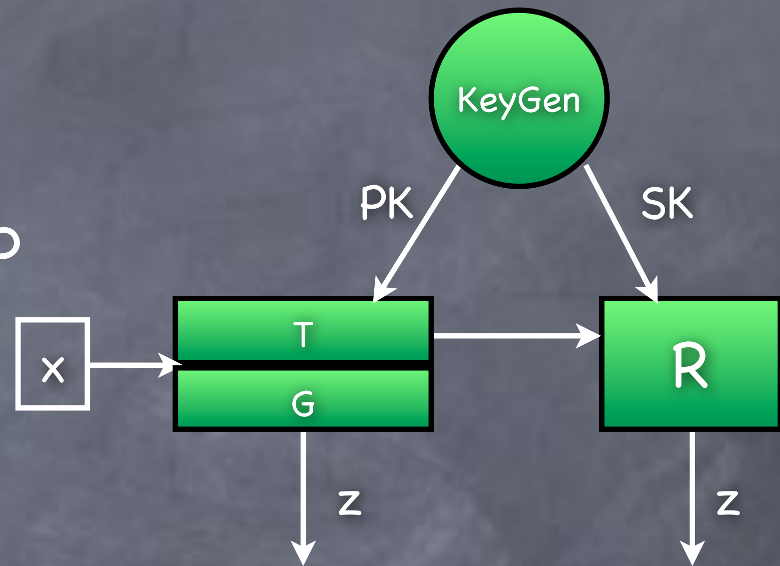

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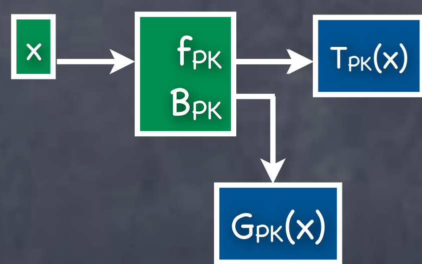

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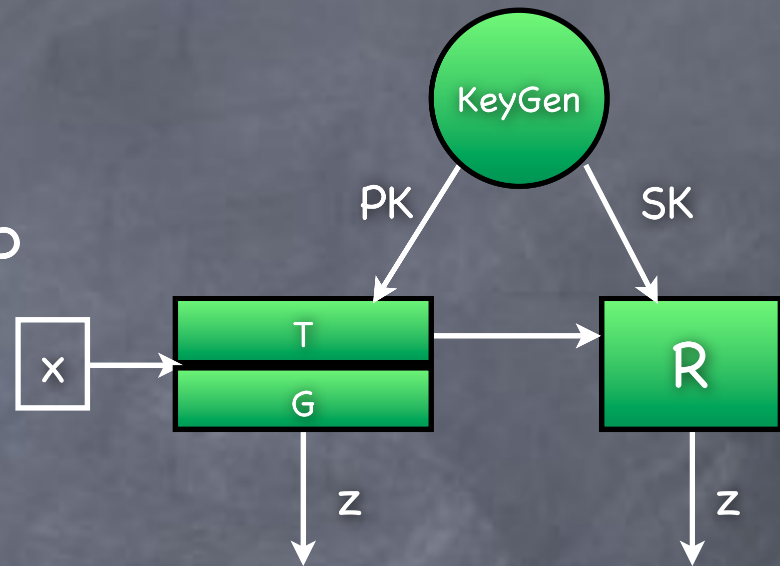

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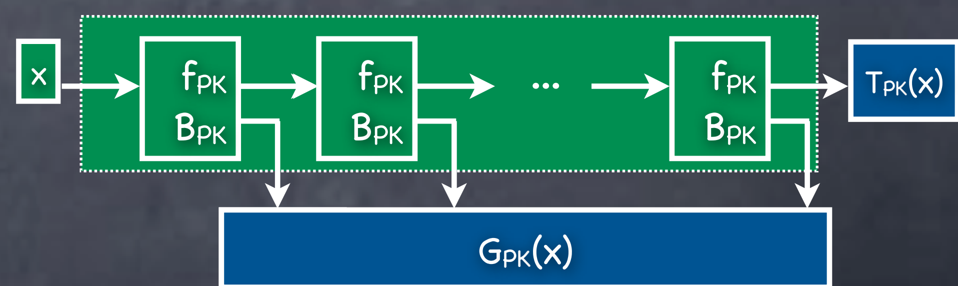

Candidate TOWPs

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    - Fact: $f_{Rabin}(.; N)$ is a permutation among quadratic residues, when P, Q are safe primes

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see handout

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