CS598JHM: **Advanced NLP** (Spring '10)

More on EM and variational inference

Julia Hockenmaier
juliahmr@illinois.edu
3324 Siebel Center

http://www.cs.uiuc.edu/class/sp10/cs598jhm

Today

- The connection between EM and variational inference

- Exponential families

Bishop (2006) sections 2.4, 9.3, 9.4, 10.1, 10.4

Exponential Families

Exponential families

- Most parametric distributions that we've seen so far belong to the exponential family of distributions

Distributions in the exponential family are nice because:

- They have **conjugate priors** (other distributions generally don't)

- The likelihood and posterior
can be expressed in terms of **sufficient statistics**

Definition

The **exponential family of distributions** over x

(x can be scalar or vector; discrete or continuous)

given the "natural" parameters $\boldsymbol{\eta}$

is defined as the set of distributions

$$p(x|\boldsymbol{\eta}) = h(x)g(\boldsymbol{\eta})exp(\boldsymbol{\eta}^T u(x))$$

- $g(\boldsymbol{\eta})$: normalization coefficient:
  $g(\boldsymbol{\eta}) = (\int x \; exp(\boldsymbol{\eta}^T u(x)))^{-1}$
- $u(x)$: some function of x

Likelihood

Given a sequence of i.i.d observations $Y=(y_1,.., y_n)$, the likelihood $P(Y|\boldsymbol{\eta})$ is:

$$P(Y|\boldsymbol{\eta}) = \left[\prod_{i=1}^{n} h(y_i)\right] g(\boldsymbol{\eta})^n \exp\left(\boldsymbol{\eta}^T \sum_{i=1}^{n} u(y_i)\right)$$

Define a function $t(Y)$, called **sufficient statistics:**

$$t(Y) = \sum_{i=1}^{n} u(y_i)$$

Thus:

$$\begin{aligned} P(Y|\boldsymbol{\eta}) &= \left[\prod_{i=1}^{n} h(y_i)\right] g(\boldsymbol{\eta})^n \exp\left(\boldsymbol{\eta}^T t(Y)\right) \\ &\propto g(\boldsymbol{\eta})^n \exp\left(\boldsymbol{\eta}^T t(Y)\right) \end{aligned}$$

Conjugate priors

It is straightforward to define a conjugate prior for members of the exponential family:

$$
\begin{aligned}
\textbf{Likelihood} \quad P(Y|\boldsymbol{\eta}) &\propto g(\boldsymbol{\eta})^n e^{\boldsymbol{\eta}^T t(Y)} \\
\textbf{Prior} \quad P(\boldsymbol{\eta}) &\propto g(\boldsymbol{\eta})^\mu e^{\boldsymbol{\eta}^T \nu} \\
\textbf{Posterior} \quad P(\boldsymbol{\eta}|Y) &\propto P(\boldsymbol{\eta})P(Y|\boldsymbol{\eta}) \\
&= g(\boldsymbol{\eta})^{\mu+n} e^{\boldsymbol{\eta}^T(\nu+t(Y))}
\end{aligned}
$$

Expectation Maximization revisited

The EM algorithm

The goal of EM: Find the maximum likelihood solution for a model consisting of parameters θ, given observed (incomplete) data **X** and latent variables **Z**

$$\ln p(\mathbf{X}|\theta) \quad = \quad \ln \left\{ \sum_{\mathbf{Z}} p(\mathbf{X}, \mathbf{Z}|\theta) \right\}$$

Note: Even if complete likelihood $p(\boldsymbol{X},\boldsymbol{Z} \mid \theta)$ is in exponential family, incomplete likelihood $p(\boldsymbol{X} \mid \theta)$ may not be.

We just have **incomplete data X**, so don't know $p(\boldsymbol{X},\boldsymbol{Z} \mid \theta)$.
We can only infer **Z** from posterior $p(\boldsymbol{Z} \mid \boldsymbol{X},\theta)$.
We will compute the **expectation** of $p(\boldsymbol{X},\boldsymbol{Z} \mid \theta)$ wrt. $p(\boldsymbol{Z} \mid \boldsymbol{X}, \theta)$

The EM algorithm

1. **Initialization:** Choose initial θ^{old}

2. **Expectation step:**
   Compute **posterior of the latent variables** $p(\mathbf{Z} \mid \mathbf{X}, \theta^{old})$

3. **Maximization step:**
   Find θ^{new} which **maximize the expected log-likelihood of the joint** $p(\mathbf{Z},\mathbf{X} \mid \theta^{new})$ under $p(\mathbf{Z} \mid \mathbf{X}, \theta^{old})$:

$$\theta^{new} = \arg\max_{\theta} \sum_{\mathbf{Z}} p(\mathbf{Z}|\mathbf{X}, \theta^{old}) \ln p(\mathbf{X}, \mathbf{Z}|\theta)$$

4. **Check for convergence.**
   Stop, or set $\theta^{old} := \theta^{new}$ and go to 2.

Another view of EM

We want to maximize $$\ln p(\mathbf{X}|\theta) = \ln \left\{ \sum_{\mathbf{Z}} p(\mathbf{X}, \mathbf{Z}|\theta) \right\}$$

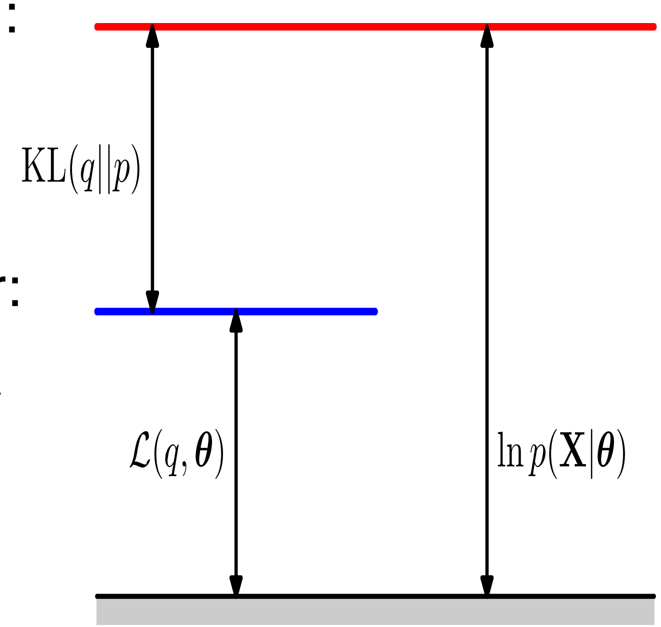
By the product rule: $$\ln p(\mathbf{X}, \mathbf{Z}|\theta) = \ln p(\mathbf{Z}|\mathbf{X}\theta) + \ln p(\mathbf{X}|\theta)$$

Define a functional of distribution $q(Z)$:

$$\mathcal{L}(q, \theta) = \sum_{\mathbf{Z}} q(\mathbf{Z}) \ln \frac{p(\mathbf{X}, \mathbf{Z}|\theta)}{q(\mathbf{Z})}$$

KL-divergence btw. $q(Z)$ and posterior:

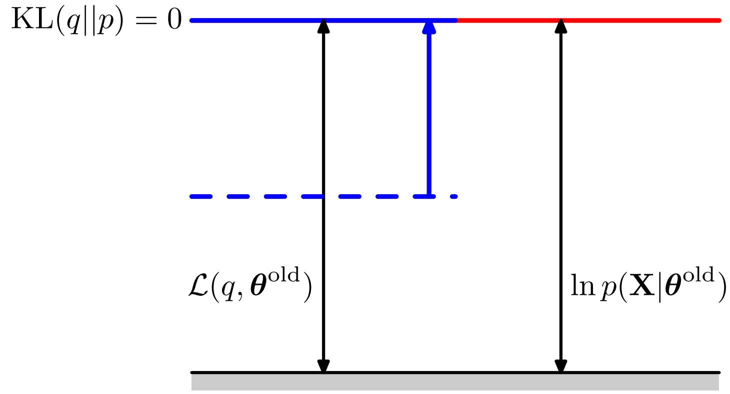
$$KL(q||p) = -\sum_{\mathbf{Z}} q(\mathbf{Z}) \ln \frac{p(\mathbf{Z}|\mathbf{X}, \theta)}{q(\mathbf{Z})}$$



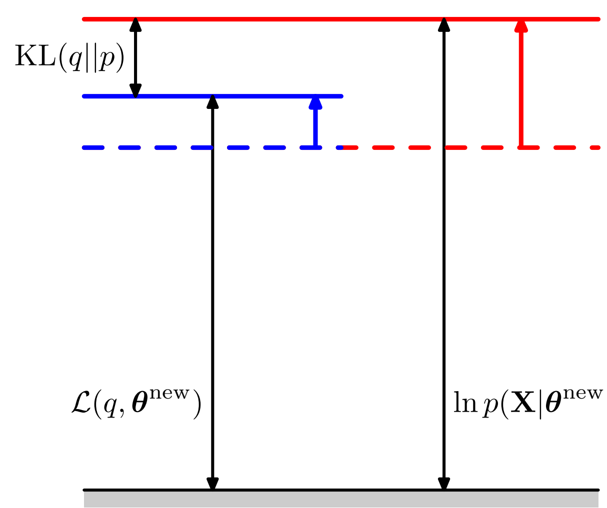
Thus $\ln p(\mathbf{X}|\theta) = \mathcal{L}(q, \theta) + KL(q||p)$

EM again...

$\mathcal{L}(q, \theta)$ is a lower bound on log-likelihood $\ln p(\boldsymbol{X} \mid \theta)$



E-step:
Maximize $\mathcal{L}(q, \theta^{old})$ wrt. $q(\boldsymbol{Z})$,
keep θ^{old} fixed.
This happens when $KL(q||p) = 0$.



M-step:
Maximize $\mathcal{L}(q, \theta^{old})$ wrt. θ,
keep $q(\boldsymbol{Z})$ fixed.
$\mathcal{L}(q, \theta)$ will increase.
Thus $\ln p(\boldsymbol{X} \mid \theta)$ will increase.
Hence, now: $KL(q||p) > 0$

Variational inference for Bayesian models

Bayesian model

- In a fully Bayesian model, all parameters θ are stochastic variables with priors.
- Now **Z** consists of latent variables and priors.
- We still want to maximize (incomplete) log-likelihood:

$$\ln p(\mathbf{X}) = \mathcal{L}(q) + KL(q||p)$$

$$\begin{aligned} \mathcal{L}(q) &= \int q(\mathbf{Z}) \ln \frac{p(\mathbf{X}, \mathbf{Z})}{q(\mathbf{Z})} d\mathbf{Z} \\ KL(q||p) &= -\int q(\mathbf{Z}) \ln \frac{p(\mathbf{Z}|\mathbf{X})}{q(\mathbf{Z})} d\mathbf{Z} \end{aligned}$$

Factorized distributions

Assume q factorizes:

$$q(\mathbf{Z}) = \prod_{i=1}^{M} q_i(\mathbf{Z}_i)$$

We still want to maximize $\mathcal{L}(q)$.

We can do this by optimizing with respect to each factor q_i in turn

$$\begin{aligned}\mathcal{L}(q) &= \int \prod_i q_i \left\{ \ln p(\mathbf{X}, \mathbf{Z}) - \sum_i \ln q_i \right\} d\mathbf{Z} \\ &= \int q_j \langle \ln p(\mathbf{X}, \mathbf{Z}) \rangle_{i \neq j} + c' \; d\mathbf{Z_j} - \int q_i \ln q_j d_{Z_j} + c\end{aligned}$$