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Price of Anarchy

Roughgarden


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Selfish People

Agents acting selfishly or even just following a dominant strategy can degrade the efficiency of a system!

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Agents acting selfishly or even just following a dominant strategy can degrade the efficiency of a system!

Just a ***little*** dictatorship might improve the situation.

Assumption!

In many mechanisms, some information is privately held by each of the players

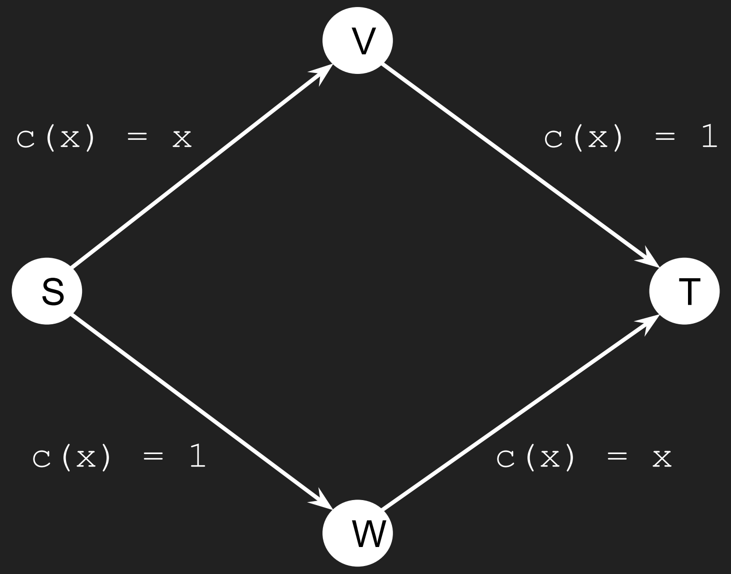
- Single-item auction: bidder preferences and valuation of good being auctioned

But, for this work, consider **only** games of public information!

Selfish Routing

- Both routes are the same, so equilibrium will result in ½ split between two paths

- Each path is 1+x ⇒ travel time is 3/2 for everyone



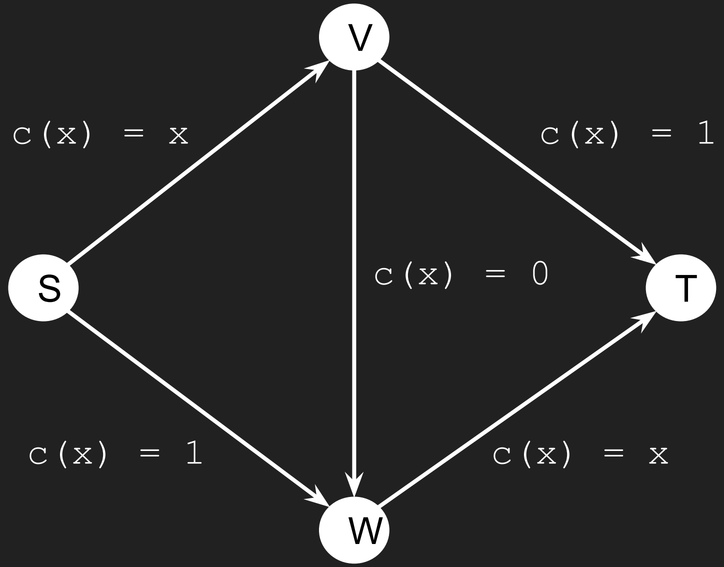
Selfish Routing

Original: 3/2 travel time

- Dominant strategy now is for everyone to take

  S → V → W → T

Why?



Selfish Routing

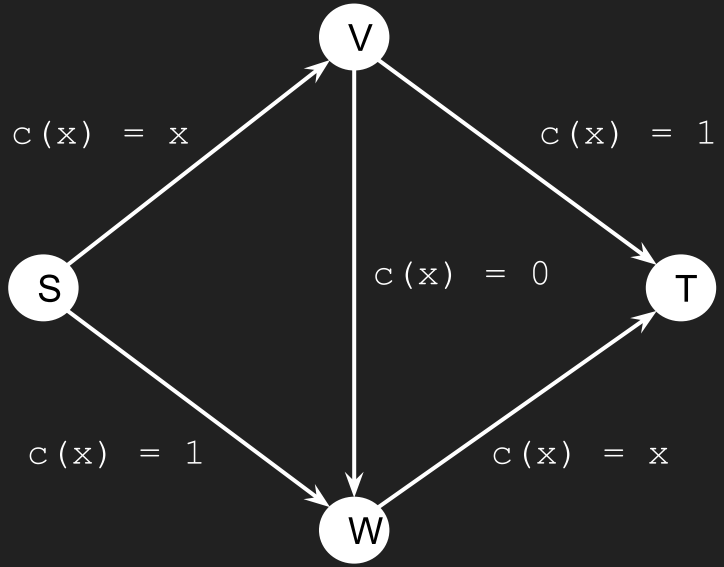
Original: 3/2 travel time

- Dominant strategy now is for everyone to take

S → V → W → T

Why?

c(s→ v→ w→ t) is *never worse* than other paths!

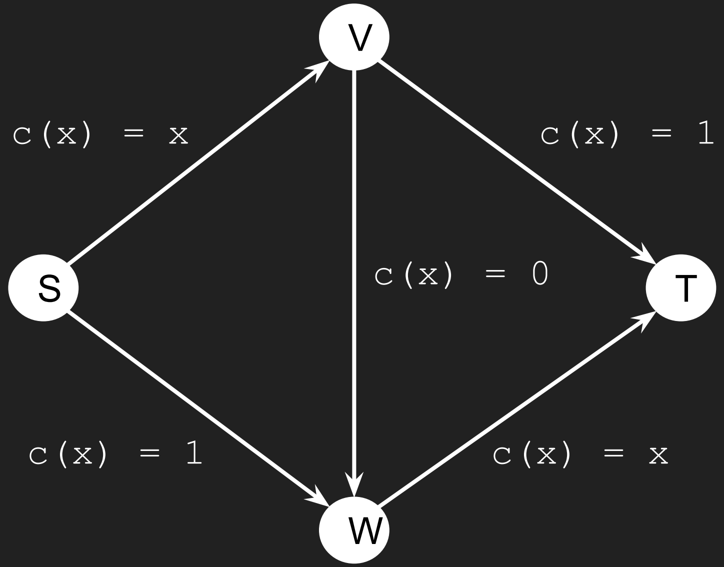


Selfish Routing

Original: 3/2 travel time

New: 2 travel time

- The minimum travel time possible is still 3/2

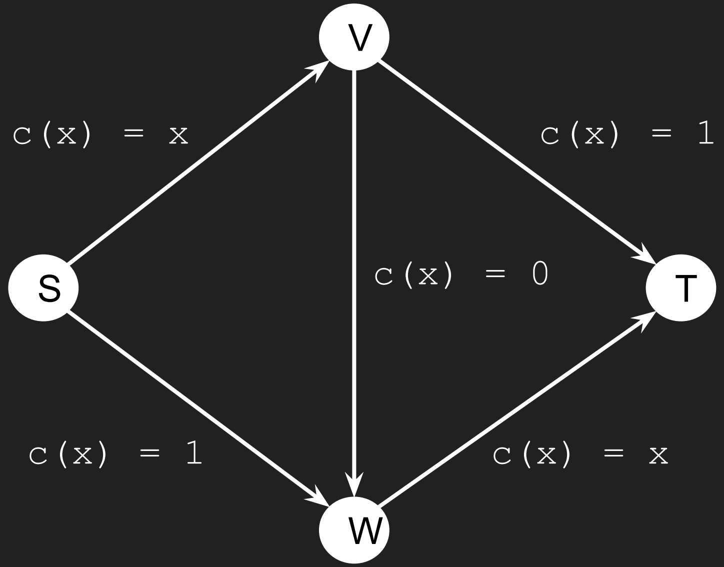


Selfish Routing

Original: 3/2 travel time

New: 2 travel time

Best: 3/2 travel time



Selfish Routing

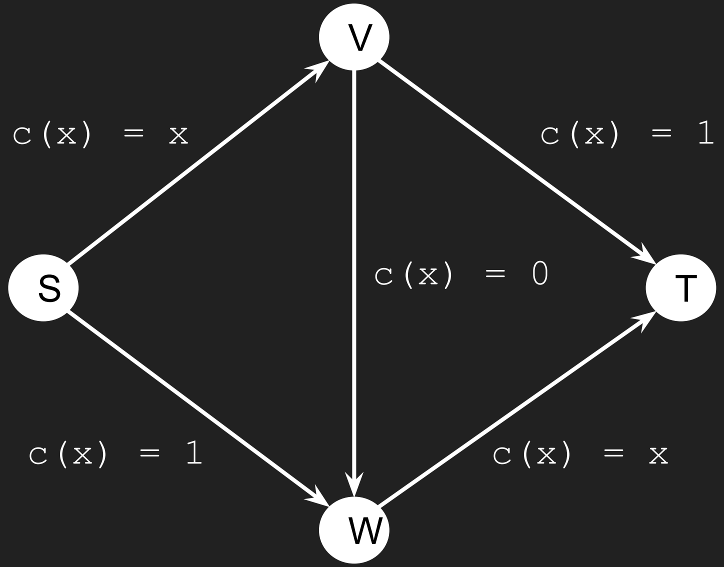
Original: 3/2 travel time

New: 2 travel time

Best: 3/2 travel time

Price of Anarchy:

(2)/(3/2) = 4/3



Prisoner's Dilemma

Only **nash equilibrium** is when both defect and tell on the other

However, the optimal solution is for both to cooperate

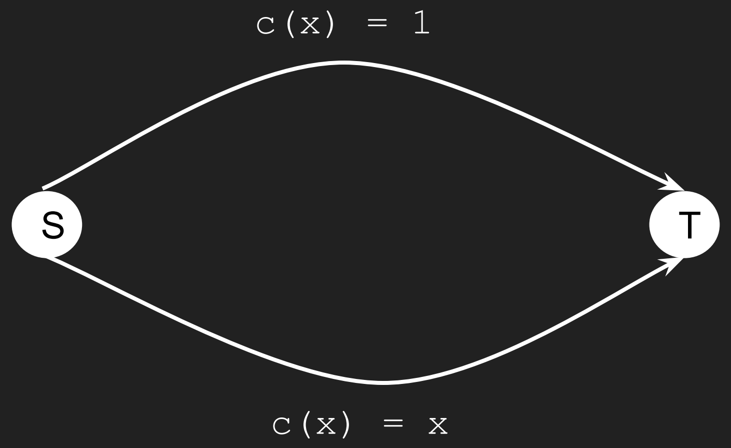
PoA = 10/2 = 5

(Cost is years in jail)

	Cooperate	Defect
Cooperate	1,1	7,0
Defect	0,7	5,5

Pigou Network

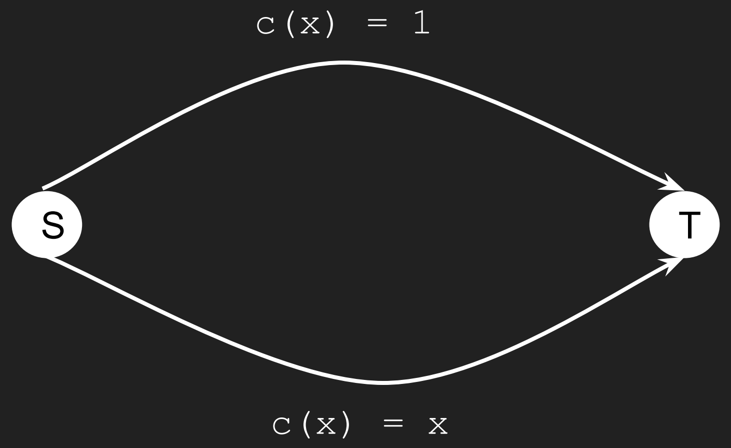
Dominant strategy?



Pigou Network

Dominant strategy? Lower edge

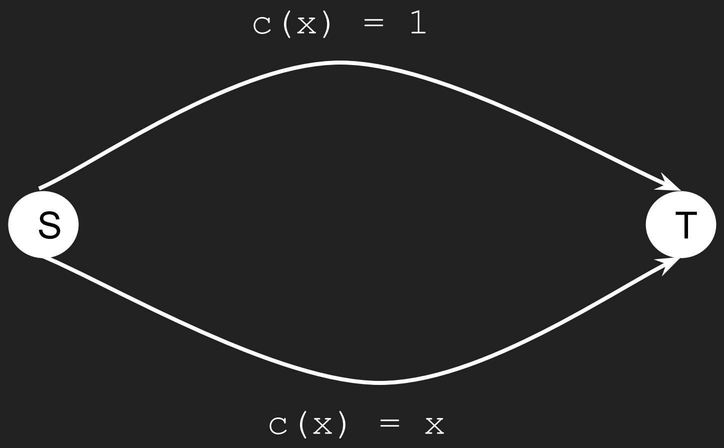
Why?



Pigou Network

Dominant strategy? Lower edge

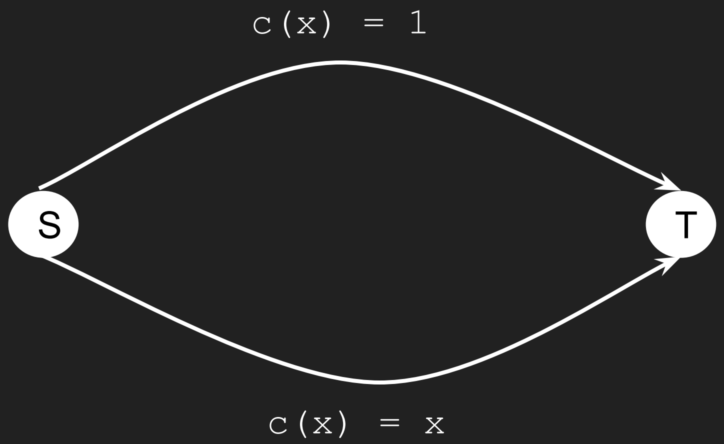
Better solution?



Pigou Network

Dominant strategy? Lower edge

Better solution? Literally anything else!

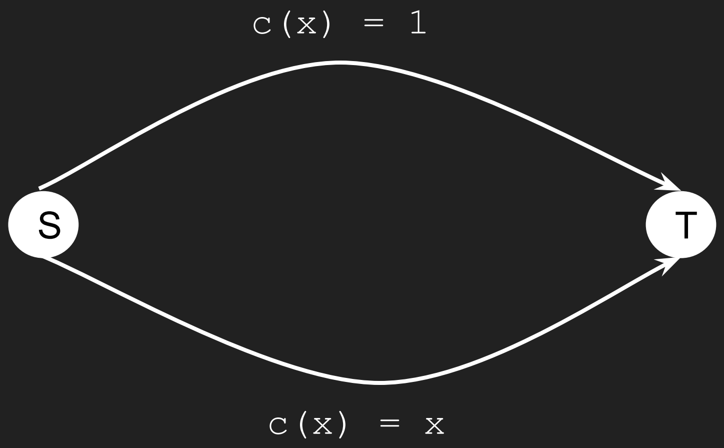


Pigou Network

Dominant strategy? Lower edge

Better solution? Literally anything else!

Best solution?

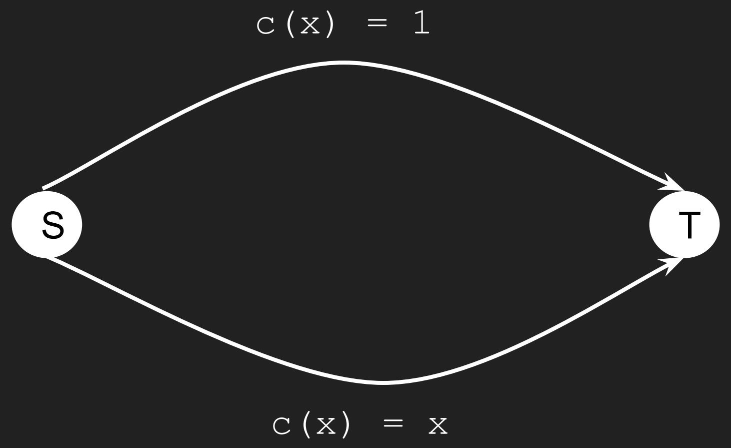


Pigou Network

Dominant strategy? Lower edge

Better solution? Literally anything else!

Best solution? Enforce a 50/50 split ⇒ ¾ travel time!



Pigou Network

Dominant strategy? Lower edge

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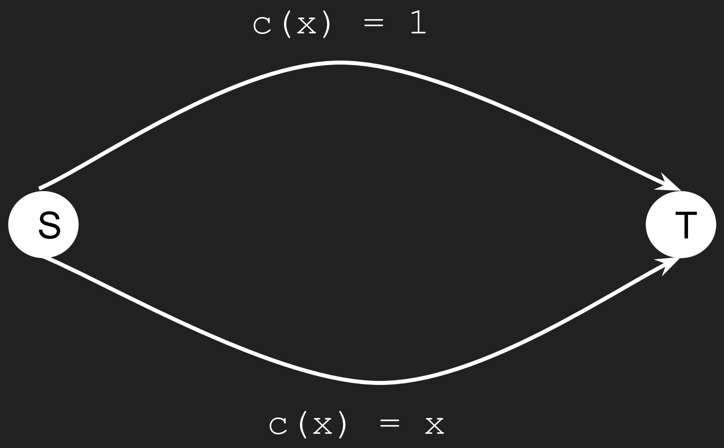
Thank your local dictator!



Pigou Network

Dominant strategy? Lower edge

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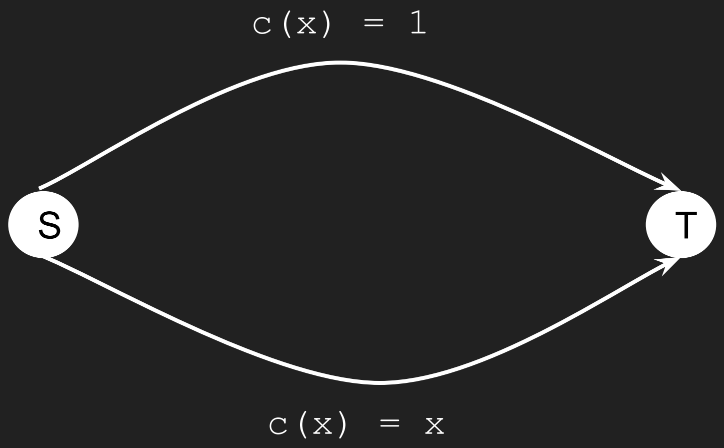
PoA:

$1/(3/4) = 4/3$



Pigou Network - non-linear cost

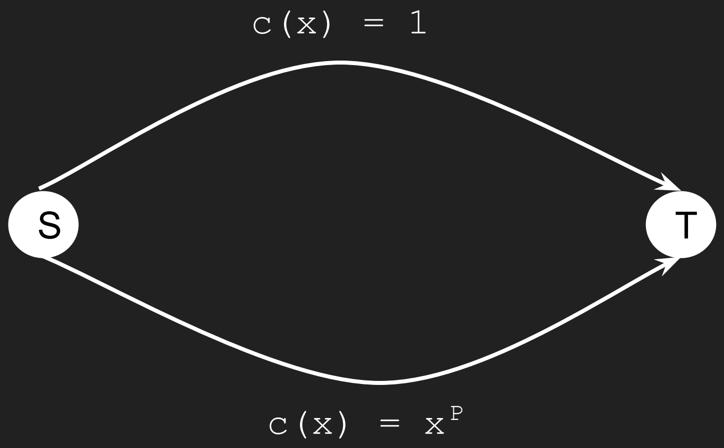
Dominant strategy? Still the lower edge ⇒ 1 travel time

Better solution? Literally anything else!

Best solution? 50/50, $p \to \infty \Rightarrow \frac{1}{2}$

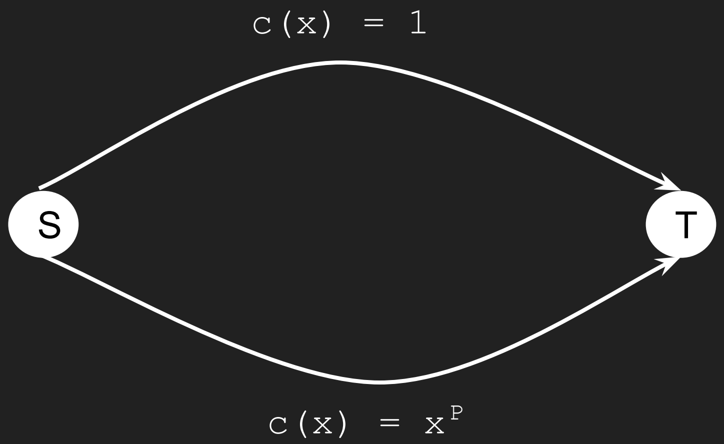
$(1-\epsilon)/\epsilon$, $p \to \infty \Rightarrow$ almost instantaneous

PoA $\to \infty$ as $p \to \infty$



Pigou Network - non-linear cost

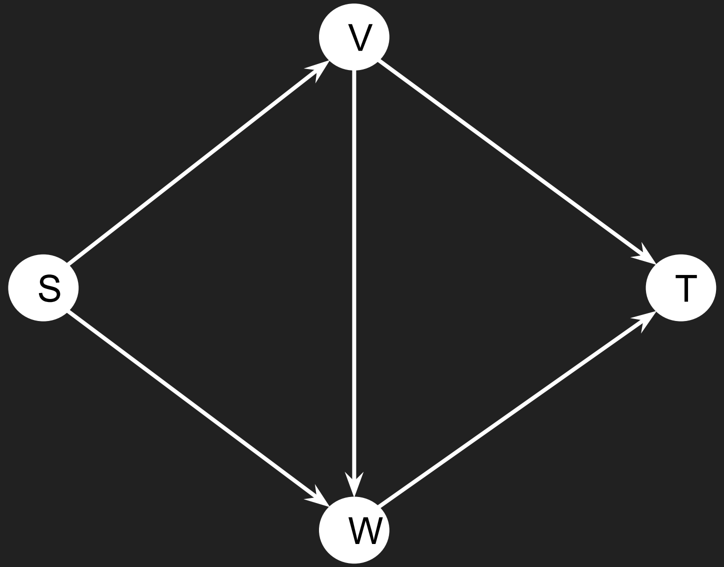
In fact: highly non-linear cost functions are the *only* obstacle to a small PoA!



Proof...

Model

- Directed graph: **G**
- One source **S** and one sink **T**
- Flow rate (traffic) of **r** travelling from **S** to **T**
- Each edge **e** has some non-negative, continuous, non-decreasing, cost function



Theorem 1 (Tight PoA Bounds)

Among all networks with cost functions in a set ***C****, the largest PoA is achieved in a Pigou-like network.*

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*Among all networks with cost functions in a set **C**, the largest PoA is achieved in a Pigou-like network.*

⇒ an upper-bound for *any* network can be derived from a Pigou-style example instead!

Theorem 1 (Tight PoA Bounds)

Among all networks with cost functions in a set ***C****, the largest PoA is achieved in a Pigou-like network.*

$C = \{c(x) = ax+b : a,b \geq 0\}$

⇒ maxPoA = 4/3 from previous example

Theorem 1 (Tight PoA Bounds)

*Among all networks with cost functions in a set **C**, the largest PoA is achieved in a Pigou-like network.*

$$C = \{c(x) = ax+b : a,b \geq 0\}$$

⇒ **maxPoA = 4/3 from previous example**

$$C = \{a_1x^d + a_2x^{d-1} + \dots + a_d : a_i \geq 0\}$$

⇒ **maxPoA = *unbounded***

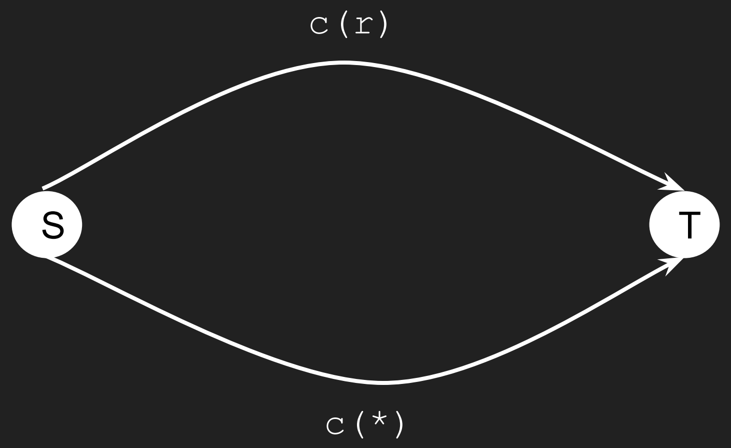
Theorem 1 (Tight PoA Bounds)

Table 1: The worst-case POA in selfish routing networks with cost functions that are polynomials with nonnegative coefficients and degree at most d.

Description	Typical Representative	Price of Anarchy
Linear	$ax + b$	$4/3$
Quadratic	$ax^2 + bx + c$	$\frac{3\sqrt{3}}{3\sqrt{3}-2} \approx 1.6$
Cubic	$ax^3 + bx^2 + cx + d$	$\frac{4\sqrt[3]{4}}{4\sqrt[3]{4}-3} \approx 1.9$
Quartic	$ax^4 + \cdots$	$\frac{5\sqrt[4]{5}}{5\sqrt[4]{5}-4} \approx 2.2$
Polynomials of degree $\leq d$	$\sum_{i=0}^{d} a_i x^i$	$\frac{(d+1)\sqrt[d]{d+1}}{(d+1)\sqrt[d]{d+1}-d} \approx \frac{d}{\ln d}$

Pigou-like Networks

- Two vertices, **s** and **t**
- Two edges from **s** to **t**
- A traffic rate **r > 0**
- A cost function **c(*)** on the first edge
- The cost function everywhere equal to **c(r)** on the second edge



Pigou-like Networks

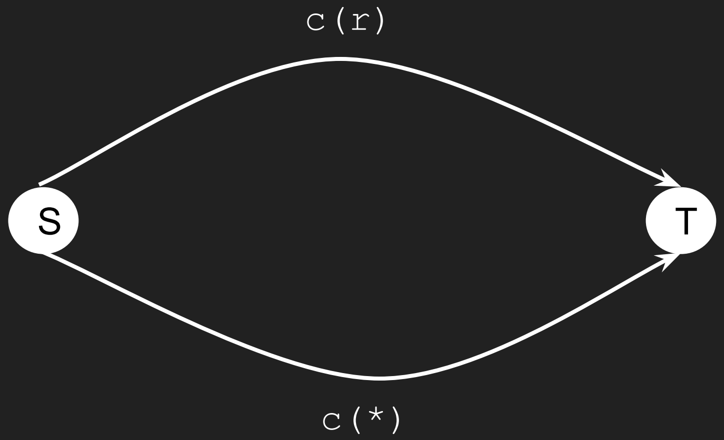
Dominant Strategy? Lower edge

⇒ **r·c(r)** travel time

Best Solution?

$$\inf_{0 \le x \le r} \{x \cdot c(x) + (r - x) \cdot c(r)\}$$

inf: greatest lower bound



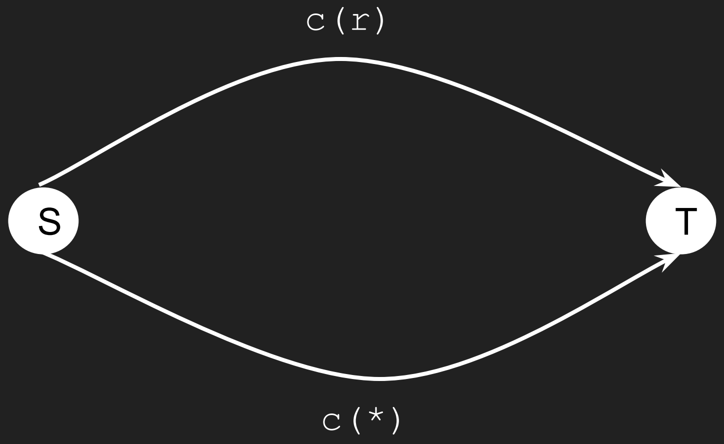
Pigou-like Networks

Dominant Strategy? Lower edge

⇒ **r·c(r)** travel time

PoA?

$$\sup_{x \geq 0} \left\{ \frac{r \cdot c(r)}{x \cdot c(x) + (r - x) \cdot c(r)} \right\}$$



sup: lowest upper bound

Pigou-like Networks

Set of cost functions $\mathcal{C}$

Pigou Bound $\alpha(\mathcal{C})$ worse PiA in a Pigou-Like network

$$\alpha(\mathcal{C}) := \sup_{c \in \mathcal{C}} \sup_{r \geq 0} \sup_{x \geq 0} \left\{ \frac{r \cdot c(r)}{x \cdot c(x) + (r - x) \cdot c(r)} \right\}$$

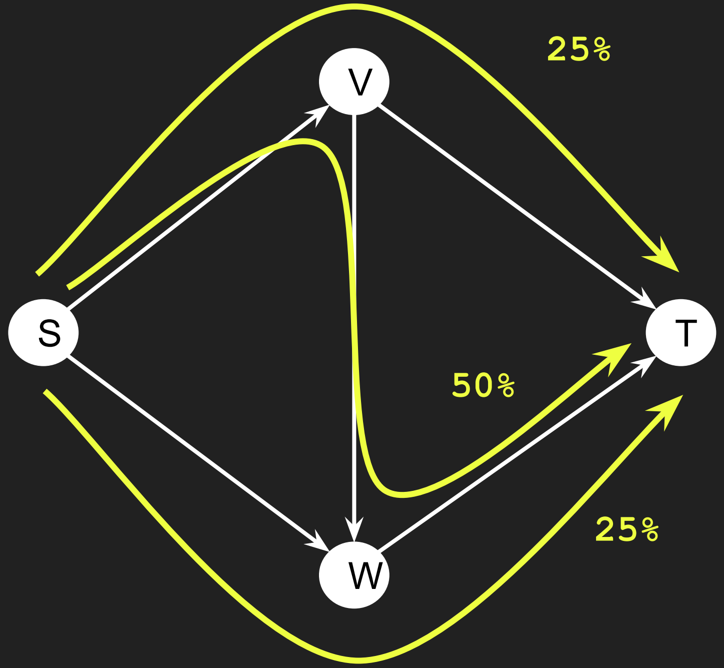
Theorem (Formal) - Right PoA Bounds

For every set $\mathcal{C}$ of cost functions and every selfish routing network with cost functions in $\mathcal{C}$, the PoA is at most $\alpha(\mathcal{C})$.

Theorem (Formal) - Right PoA Bounds

Define ***equilibrium flow*** *as travel only on shortest* S → T *paths, i.e.* $f_P > 0$ ***iff***

$$\widehat{P} \in \operatorname*{argmin}_{P \in \mathcal{P}} \Bigg\{ \underbrace{\sum_{e \in P} c_e(f_e)}_{:= c_P(f)} \Bigg\}$$



Theorem (Formal) - Right PoA Bounds

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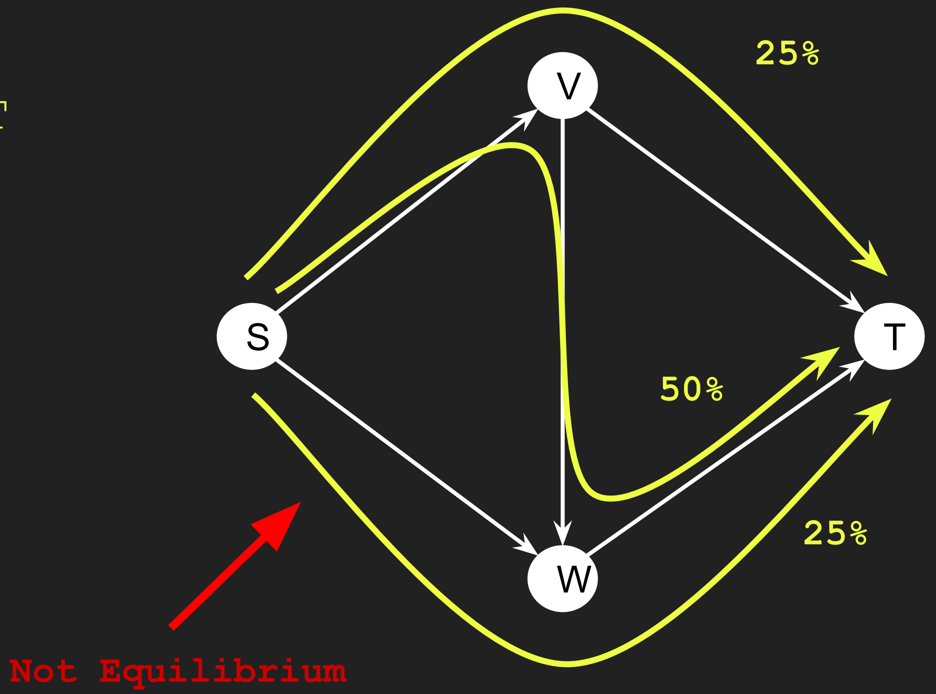

Theorem (Formal) Proof

On the board… if you would like to see it

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On the board… if you would like to see it

Preliminary:

- $C(f) = \sum_{P \in \mathcal{P}} f_P \cdot c_P(f)$ ⇐ total travel time of some flow f
- All equilibrium flows in a graph G have the same cost
- f and f^* are the equilibrium and optimal flows of a graph G

Theorem (Formal) Proof

Main Points:

1. Fixing all edge costs in the graph to be $c_e(f_e)$, their cost in the equilibrium flow f, makes it **optimal**
   a. Straightforward as equilibrium routes through shortest paths for everyone

All paths P' used by equilibrium flow have a common cost $c_{P'}(f) := L$

⇒ For all $P \in \mathcal{P}$, $c_P(f) \geq L$, i.e. equilibrium is at least as good as any other flow

Theorem (Formal) Proof

Main Points:

$\Rightarrow \sum_{P \in \mathcal{P}} (f_e^* - f_e) \cdot c_e(f_e) \geq 0$

When edge costs are frozen at equilibrium costs, no other flow f^* can be better than f

Theorem (Formal) Proof

Main Points:

2. Re-examine the Pigou-Bound to see how much better $f^{\star}$ is then f

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2. Re-examine the Pigou-Bound to see how much better f^* is then f

$$\alpha(\mathcal{C}) := \sup_{c \in \mathcal{C}} \sup_{r \geq 0} \sup_{x \geq 0} \left\{ \frac{r \cdot c(r)}{x \cdot c(x) + (r - x) \cdot c(r)} \right\}$$

Theorem (Formal) Proof

For all e∈E, substituting into Pigou-Bound yields

$$\alpha(\mathcal{C}) \geq \frac{f_e \cdot c_e(f_e)}{f_e^* \cdot c_e(f_e^*) + (f_e - f_e^*)c_e(f_e)}$$

Theorem (Formal) Proof

Summing over all edges the inequality yields:

$$C(f^*) \geq \frac{1}{\alpha(\mathcal{C})} \cdot C(f) + \underbrace{\sum_{e \in E} (f_e^* - f_e) c_e(f_e)}_{\geq 0 \text{ by } (5)} \geq \frac{C(f)}{\alpha(\mathcal{C})}$$

Theorem (Formal) Proof

$\Rightarrow$ $c(f)/c(f^*) \leq \alpha(C)$

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$\Rightarrow$ $C(f)/C(f^*) \leq \alpha(\mathcal{C})$

PoA for any graph with a set of cost functions $\mathcal{C}$ is bounded by the Pigou-bound!