CS477 Formal Software Development Methods

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DEMO

Algortihm for Proving Hoare Triples?

- Have seen in Isabelle that much of proving a Hoare triple is routine
- Will this always work?
- Why not automate the whole process?
  - Can't (always) calculate needed loop invariants
  - Can't (always) prove implications (side-conditions) in Rule of Consequence application
- Can we automate all but this?
- Yes! But how?

1. Annotate all `while` loops with needed `invariants`
2. Use routine to "roll back" post-condition to weakest precondition, gathering side-conditions as we go

- 2 called verification condition generation

Annotated Simple Imperative Language

- Give verification conditions for an annotated version of our simple imperative language
- Add a presumed invariant to each while loop

$$\begin{aligned}\langle \textit{command}\rangle \ ::=\ & \langle \textit{variable}\rangle := \langle \textit{term}\rangle\\ \mid\ & \langle \textit{command}\rangle;\ \ldots;\ \langle \textit{command}\rangle\\ \mid\ & \textit{if}\ \langle \textit{statement}\rangle\ \textit{then}\ \langle \textit{command}\rangle\ \textit{else}\ \langle \textit{command}\rangle\\ \mid\ & \textit{while}\ \langle \textit{statement}\rangle\ \textit{inv}\ \langle \textit{statement}\rangle\ \textit{do}\ \langle \textit{command}\rangle\end{aligned}$$

Example:
```
while y < n inv x = y * y
do
 x := (2 * y) + 1;
 y := y + 1
od
```


Hoare Logic for Annotated Programs

Assingment Rule
$$\frac{}{\{\!|P[e/x]|\!\}\ x\ :=\ e\ \{\!|P|\!\}}$$

Rule of Consequence
$$\frac{P \Rightarrow P' \quad \{\!|P'|\!\}\ C\ \{\!|Q'|\!\} \quad Q' \Rightarrow Q}{\{\!|P|\!\}\ C\ \{\!|Q|\!\}}$$

Sequencing Rule
$$\frac{\{\!|P|\!\}\ C_1\ \{\!|Q|\!\} \quad \{\!|Q|\!\}\ C_2\ \{\!|R|\!\}}{\{\!|P|\!\}\ C_1;\ C_2\ \{\!|R|\!\}}$$

If Then Else Rule
$$\frac{\{\!|P \wedge B|\!\}\ C_1\ \{\!|Q|\!\} \quad \{\!|P \wedge \neg B|\!\}\ C_2\ \{\!|Q|\!\}}{\{\!|P|\!\}\ \textit{if}\ B\ \textit{then}\ C_1\ \textit{else}\ C-2\ \{\!|Q|\!\}}$$

While Rule
$$\frac{\{\!|P \wedge B|\!\}\ C\ \{\!|P|\!\}}{\{\!|P|\!\}\ \textit{while}\ B\ \textit{inv}\ P\ \textit{do}\ C\ \{\!|P \wedge \neg B|\!\}}$$

Relation Between Two Languages

- Hoare Logic for Simple Imperative Programs and Hoare Logic to Annotated Programs almost the same
- What it precise relationship?
- First need precise relation between the two languages

Definition

$\mathrm{strip}(v := e)\ =\ v := e$
$\mathrm{strip}(C_1\ ;\ C_2)\ =\ \mathrm{strip}(C_1)\ ;\ \mathrm{strip}(C_2)$
$\mathrm{strip}(\texttt{if}\ B\ \texttt{then}\ C_1\ \texttt{else}\ C_2\ \texttt{fi})\ =$
$\quad \texttt{if}\ B\ \texttt{then}\ \mathrm{strip}(C_1)\ \texttt{else}\ \mathrm{strip}(C_2)\ \texttt{fi}$
$\mathrm{strip}(\texttt{while}\ B\ \texttt{inv}\ P\ \texttt{do}\ C\ \texttt{od})\ =\ \texttt{while}\ B\ \texttt{do}\ \mathrm{strip}(C)\ \texttt{od}$

- We recursively remove all invariant annotations from all `while` loops

Relation Between Two Hoare Logics

Theorem

For all pre- and post-conditions P and Q, and annotated programs C, if $\{|P|\}\ C\ \{|Q|\}$, then $\{P\}\ strip(C)\ \{Q\}$.

Proof.

(Sketch) Use rule induction on proof of $\{|P|\}\ C\ \{|Q|\}$; in case of While Rule, erase invariant □

Relation Between Two Hoare Logics

Theorem

For all pre- and post-conditions P and Q, and unannotated programs C, if $\{P\}\ C\ \{Q\}$, then there exists an annotated program S such that $C = strip(()S)$ and $\{|P|\}\ S\ \{|Q|\}$.

Proof.

(Sketch) Use rule induction on proof of $\{P\}\ C\ \{Q\}$; in case of While Rule, add invariant from precondition as invariant to command. □

Weakest Precondition

Question: Given post-condition Q, and annotated program C, what is the most general pre-condition P such that $\{|P|\}\ C\ \{|Q|\}$?

Answer: Weakest Precondition

Definition

$\mathrm{wp}\ (x := e)\ Q = Q[x \Rightarrow e]$
$\mathrm{wp}\ (C_1; C_2)\ Q = \mathrm{wp}\ C_1\ (\mathrm{wp}\ C_2\ Q)$
$\mathrm{wp}\ (\texttt{if}\ B\ \texttt{then}\ C_1\ \texttt{else}\ C_2\ \texttt{fi})\ Q =$
$\quad (B \wedge (\mathrm{wp}\ C_1\ Q)) \vee ((\neg B) \wedge (\mathrm{wp}\ C_2\ Q))$
$\mathrm{wp}\ (\texttt{while}\ B\ \texttt{inv}\ P\ \texttt{do}\ C\ \texttt{od})\ Q = P$

Assumes, without verifying, that P is the correct invariant

Weakest Justification

Weakest in weakest precondition means any other valid precondition implies it:

Theorem

For all annotated programs C, and pre- and post-conditions P and Q, if $\{|P|\}\ C\ \{|Q|\}$ then $P \Rightarrow wp\ C\ Q$.

- Proof somewhat complicated
- Uses induction structure of C
- In each case, want to assert triple proof must have used rule for that construct (e.g. Sequence Rule for sequences)
- Can't because of Rule Of Consequence
- Must induct on proof (rule induction) - in each case
- Uses:

Lemma

$\forall C\ P\ Q.\ (P \Rightarrow Q) \Rightarrow (wp\ C\ P \Rightarrow wp\ C\ Q)$

What About Precondition?

Question: Do we have $\{|\mathrm{wp}\ C\ Q|\}\ C\ \{|Q|\}$?

Answer: Not always - need to check while-loop side-conditions – verification conditions

Question: How to calculate verification conditions?

Definition

$\mathrm{vcg}\ (x := e)\ Q = \mathrm{true}$
$\mathrm{vcg}\ (C_1; C_2)\ Q = (\mathrm{vcg}\ C_1\ (\mathrm{wp}\ C_2\ Q)) \wedge (\mathrm{vcg}\ C_2\ Q)$
$\mathrm{vcg}\ (\texttt{if}\ B\ \texttt{then}\ C_1\ \texttt{else}\ C_2\ \texttt{fi})\ Q = (\mathrm{vcg}\ C_1\ Q) \wedge (\mathrm{vcg}\ C_2\ Q)$
$\mathrm{vcg}\ (\texttt{while}\ B\ \texttt{inv}\ P\ \texttt{do}\ C\ \texttt{od})\ Q =$
$\quad ((P \wedge B) \Rightarrow (\mathrm{wp}\ C\ P)) \wedge (\mathrm{vcg}\ C\ P) \wedge ((P \wedge (\neg B)) \Rightarrow Q)$

Verification Condition Guarantees wp Precondition

Theorem

$vcg\ C\ Q \Rightarrow \{|wp\ C\ Q|\}\ C\ \{|Q|\}$

Proof.

(Sketch)

- Induct on structure of C
- For each case, wind back as we did in specific examples:
  - Assignment: $wp\ C\ Q$ exactly what is needed for Assignment Axiom
  - Sequence: Follows from inductive hypotheses, all elim, and modus ponens
  - If_Then_Else: Need to use Precondition Strengthening with each branch of conditional; wp and inductive hypotheses give the needed side conditions
  - While: Need to use Postcondition Weakening, While Rule and Precondition Strengthening

□

Verification Condition Guarantees wp Precondition

Corollary

$((P \Rightarrow wp\ C\ Q) \wedge (vcg\ C\ Q)) \Rightarrow \{\!|P|\!\}\ C\ \{\!|Q|\!\}$

This amounts to a method for proving Hoare triple $\{P\}\ C\ \{Q\}$:

1. Annotate program with loop invariants (reduces to showing $\{\!|P|\!\}\ C\ \{\!|Q|\!\}$
2. Calculate $\mathrm{wp}\ C\ Q$ and $\mathrm{vcg}\ C\ Q$ (automated)
3. Prove $P \Rightarrow \mathrm{wp}\ C\ Q$ and $\mathrm{vcg}\ C\ Q$

Basic outline of interaction with Boogie: Human does 1, Boogie does 2, Z3 / Simplify / Isabelle + human / ... does 3

For more infomation

- http://research.microsoft.com/en-us/projects/boogie/
- http://research.microsoft.com/en-us/um/people/moskal/pdf/hol-boogie.pdf
- http://www.cl.cam.ac.uk/research/hvg/Isabelle/dist/library/HOL/HOL-Hoare/index.html