CS477 Formal Software Development Methods

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Slides based in part on previous lectures by Mahesh Vishwanathan, and by Gul Agha

February 8, 2013

Proof Rules

Will give Sequent version of Natural Deduction rules
All rules from Propositional Logic included

$$\frac{\Gamma \vdash \psi[t/x]}{\Gamma \vdash \exists x.\psi}\ \text{Ex I}$$

$$\frac{\Gamma \vdash \exists x.\psi \quad \Gamma \cup \{(\psi[y/x])\} \vdash \varphi}{\Gamma \vdash \varphi}\ \text{Ex E}$$

provided

$y \notin fv(\varphi) \cup (fv(\psi) \setminus \{x\}) \cup \bigcup_{\psi' \in \Gamma} fv(\psi')$

$$\frac{\Gamma \vdash \psi[y/x]}{\Gamma \vdash \forall x.\psi}\ \text{All I}$$

$$\frac{\Gamma \vdash \forall x.\psi \quad \Gamma \cup \{\psi[t/x]\} \vdash \varphi}{\Gamma \vdash \varphi}\ \text{All E}$$

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$y \notin (fv(\psi) \setminus \{x\}) \cup \bigcup_{\psi' \in \Gamma} fv(\psi')$

Example

Show

$$\frac{}{\{\ \} \vdash (\exists x.\, \forall y.\, x \leq y) \Rightarrow (\forall x.\, \exists y.\, y \leq x)}$$

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$$\dfrac{\dfrac{}{\{(\exists x.\, \forall y.\, x \leq y)\} \vdash \forall x.\, \exists y.\, y \leq x}}{\{\,\} \vdash (\exists x.\, \forall y.\, x \leq y) \Rightarrow (\forall x.\, \exists y.\, y \leq x)}\ \text{Imp I}$$

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Example of Failure

Let's try to show

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Floyd-Hoare Logic

- Also called Axiomatic Semantics
- Based on formal logic (first order predicate calculus)
- Logical system built from axioms and inference rules
- Mainly suited to simple imperative programming languages
- Ideas applicable quite broadly

Floyd-Hoare Logic

- Used to formally prove a property (post-condition) of the state (the values of the program variables) after the execution of program, assuming another property (pre-condition) of the state holds before execution

Floyd-Hoare Logic

- Goal: Derive statements of form

$$\{P\}\ C\ \{Q\}$$

  - P, Q logical statements about state, P precondition, Q postcondition, C program

- Example:

$$\{x = 1\}\ x := x + 1\ \{x = 2\}$$

Floyd-Hoare Logic

- **Approach:** For each type of language statement, give an axiom or inference rule stating how to derive assertions of form

$$\{P\}\ C\ \{Q\}$$

  where C is a statement of that type
- Compose axioms and inference rules to build proofs for complex programs

Partial vs Total Correctness

- An expression $\{P\}\ C\ \{Q\}$ is a partial correctness statement
- For total correctness must also prove that C terminates (i.e. doesnt run forever)
  - Written: $[P]\ C\ [Q]$
- Will only consider partial correctness here

Simple Imperative Language

- We will give rules for simple imperative language

$$\begin{aligned}\langle command\rangle \;::=\; & \langle variable\rangle := \langle term\rangle \\ & \mid \langle command\rangle;\ \ldots;\ \langle command\rangle \\ & \mid \mathit{if}\ \langle statement\rangle\ \mathit{then}\ \langle command\rangle\ \mathit{else}\ \langle command\rangle \\ & \mid \mathit{while}\ \langle statement\rangle\ \mathit{do}\ \langle command\rangle\end{aligned}$$

- Could add more features, like for-loops

Substitution

- Notation: $P[e/v]$ (sometimes $P[v \rightarrow e]$)
- Meaning: Replace every v in P by e
- Example:

$$(x + 2)[y - 1/x] = ((y - 1) + 2)$$

The Assingment Rule

$$\frac{}{\{P[e/x]\}\ x\ :=\ e\ \{P\}}$$

Example:

$$\frac{}{\{\quad ?\quad \}\ x\ :=\ y\ \{\ x\ = 2\ \}}$$

The Assingment Rule

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The Assingment Rule

$$\frac{}{\{P[e/x]\}\ x\ :=\ e\ \{P\}}$$

Examples:

$$\frac{}{\{y = 2\}\ x\ :=\ y\ \{x = 2\}}$$

$$\frac{}{\{y = 2\}\ x\ :=\ 2\ \{y = x\}}$$

$$\frac{}{\{x + 1 = n + 1\}\ x\ :=\ x + 1\ \{x = n + 1\}}$$

$$\frac{}{\{2 = 2\}\ x\ :=\ 2\ \{x = 2\}}$$

The Assignment Rule – Your Turn

- What is the weakest precondition of

$$x := x + y \{ x + y = wx \}?$$

$$\begin{array}{c} \{ \quad ? \quad \} \\ x := x + y \\ \{ x + y = wx \} \end{array}$$

The Assignment Rule – Your Turn

- What is the weakest precondition of

$$x := x + y \{ x + y = wx \}?$$

$$\{ (x + y) + y = w(x + y) \}$$
$$x := x + y$$
$$\{ x + y = wx \}$$

Precondition Strengthening

$$\frac{(P \Rightarrow P')\{P'\}\ C\ \{Q\}}{\{P\}\ C\ \{Q\}}$$

- Meaning: If we can show that P implies P' (*i.e.* $(P \Rightarrow P')$ and we can show that $\{P\}\ C\ \{Q\}$, then we know that $\{P\}\ C\ \{Q\}$
- P is stronger than P' means $P \Rightarrow P'$

Precondition Strengthening

- Examples:

$$\frac{x = 3 \Rightarrow x < 7 \quad \{x < 7\}\ x\ :=\ x + 3\ \{x < 10\}}{\{x = 3\}\ x\ :=\ x + 3\ \{x < 10\}}$$

$$\frac{\mathit{True} \Rightarrow (2 = 2) \quad \{2 = 2\}\ x\ :=\ 2\ \{x = 2\}}{\{\mathit{True}\}\ x\ :=\ 2\ \{x = 2\}}$$

$$\frac{x = n \Rightarrow x + 1 = n + 1 \qquad \{x + 1 = n + 1\}\ x\ :=\ x + 1\ \{x = n + 1\}}{\{x = n\}\ x\ :=\ x + 1\ \{x = n + 1\}}$$

Which Inferences Are Correct?

$$\frac{\{x > 0 \wedge x < 5\}\ x\ :=\ x * x\ \{x < 25\}}{\{x = 3\}\ x\ :=\ x * x\ \{x < 25\}}$$

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