#### CS477 Formal Software Development Methods

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Slides based in part on previous lectures by Mahesh Vishwanathan, and by Gul Agha

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# Logical Equivalence a Structural Congruence

#### Theorem

Logical equivalence is a structural congruence. That is, if  $p \equiv p'$  and  $q \equiv q'$  then

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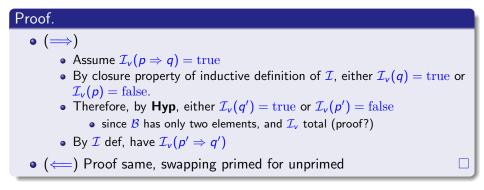
# Logical Equivalence a Structural Congruence

#### Proof.

- Assume  $p \equiv p'$  and  $q \equiv q'$
- Hyp: Then for all valuations  $v, v \models p$  iff  $v \models p'$  and  $v \models q$  iff  $v \models q'$ , i.e.  $\mathcal{I}_v(p) = \text{true}$  iff  $\mathcal{I}_v(p') = \text{true}$  and  $\mathcal{I}_v(q) = \text{true}$  iff  $\mathcal{I}_v(q') = \text{true}$
- Case 4: Show  $p \Rightarrow q \equiv p' \Rightarrow q'$ 
  - Other cases done same way
- Need to show for all v,  $\mathcal{I}_v(p \Rightarrow q) = \text{true iff } \mathcal{I}_v(p' \Rightarrow q') = \text{true}$
- Fix v
- Need to show if  $\mathcal{I}_{\nu}(p \Rightarrow q) = \text{true}$  then  $\mathcal{I}_{\nu}(p' \Rightarrow q') = \text{true}$ , and if  $\mathcal{I}_{\nu}(p' \Rightarrow q') = \text{true}$  then  $\mathcal{I}_{\nu}(p \Rightarrow q) = \text{true}$

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# Logical Equivalence a Structural Congruence



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# Non-standard Model of Propositional Logic

Other models possible Example:

- $C = \{ true, false, \bot \}$
- Valuations w assign values in C to propositional atoms
- If  $\mathcal{J}_w(p) = \bot$  then  $\mathcal{J}_w(\neg p) = \bot$ , otherwise same as for  $\mathcal{I}$
- $\mathcal{J}_w(p) = \bot$  or  $\mathcal{J}_w(q) = \bot$  then  $\mathcal{J}_w(\neg p) = \bot$ ,  $\mathcal{J}_w(p \land q) = \bot$ ,  $\mathcal{J}_w(p \lor q) = \bot$ ,  $\mathcal{J}_w(p \Rightarrow q) = \bot$ , and  $\mathcal{J}_w(p \Leftrightarrow q) = \bot$ ; otherwise same as for  $\mathcal{I}$
- Note:  $A \lor \neg A \not\equiv \mathsf{T}$

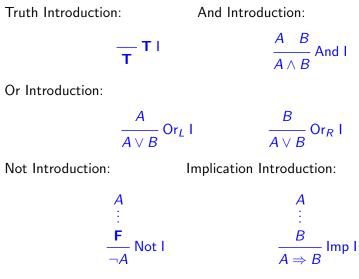
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# Proofs in Propositional Logic

- Natural Deduction proofs are trees with nodes that are inference rules
- Inference rule has hypotheses and conclusion
- Conclusion a single proposition
- Hypotheses zero or more propositions, possibly with hypotheses
- Two main kinds of inference rules:
  - Introduction says how to conclude proposition made from connective is true
  - Eliminations says how to use a proposition made from connective to prove result
- Inference rules associated with connectives
- Rule with no hypotheses called an axiom

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## Introduction Rules



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 $A \Rightarrow (B \Rightarrow (A \land B))$ 

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$$\frac{B \Rightarrow (A \land B)}{A \Rightarrow (B \Rightarrow (A \land B))} \text{ Imp I}$$

$$\frac{A \quad B}{A \wedge B}$$

$$\frac{B \Rightarrow (A \wedge B)}{A \Rightarrow (B \Rightarrow (A \wedge B))} \text{ Imp I}$$

$$\frac{A \quad B}{A \wedge B} \text{And I}$$

$$\frac{B \Rightarrow (A \wedge B)}{A \Rightarrow (B \Rightarrow (A \wedge B))} \text{Imp I}$$
Imp I

$$\frac{A \quad B}{A \wedge B} \text{And I}$$

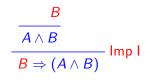
$$\frac{B \Rightarrow (A \wedge B)}{A \Rightarrow (B \Rightarrow (A \wedge B))} \text{Imp I}$$

$$\frac{B \Rightarrow (A \wedge B)}{A \Rightarrow (B \Rightarrow (A \wedge B))} \text{Imp I}$$

• All assumptions discharged; proof complete

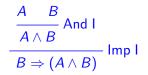
 $B \Rightarrow (A \land B)$ 

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$$\frac{A \quad B}{A \land B} \text{And I}$$
$$\frac{B \Rightarrow (A \land B)}{B \Rightarrow (A \land B)} \text{Imp I}$$

$$\frac{\frac{A? \quad B}{A \land B}}{B \Rightarrow (A \land B)} \text{ Imp I}$$



- Closed proofs must discharge all hypotheses
- Otherwise have theorem relative to / under undischarged hypotheses
- Here have proved "Assuming A, we have  $B \Rightarrow (A \land B)$

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 $A \Rightarrow (A \land A)$ 

## Discharging Hypothesis

$$\frac{A \quad A}{A \land A} \text{And I}$$

$$\frac{A}{A \Rightarrow (A \land A)} \text{Imp I}$$

$$\frac{A \quad A}{A \land A} \text{And I}$$
$$\frac{A}{A \Rightarrow (A \land A)} \text{Imp I}$$

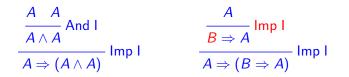
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$$\frac{A \quad A}{A \land A} \text{And I}}{A \Rightarrow (A \land A)} \text{Imp I} \qquad \qquad \overline{A \Rightarrow (B \Rightarrow A)}$$

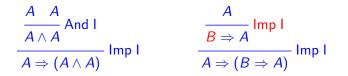
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- Imp I (and other rules discharging assumptions) may discharge multiple instance of hypothesis
- Or may discharge none at all
- Every assumption instance discharged only once

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### Your Turn

#### $A \Rightarrow (A \lor B)$

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## **Elimination Rules**

- So far, have rules to "introduce" logical connectives into propositions
- No rules for how to "use" logical connectives
  - No assumptions with logical connectives
- Need "elimination" rules
- Example: Can't prove

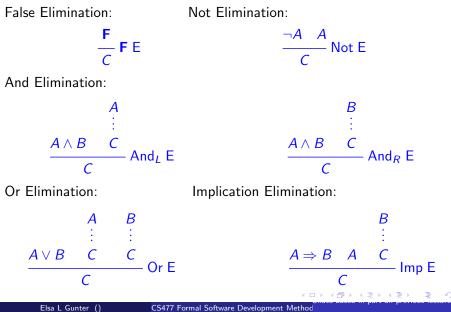
#### $(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))$

with what we have so far

- Elimination rules assume assumption with a connective; have general conclusion
  - Generally needs additional hypotheses

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## **Elimination Rules**



#### $(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))$

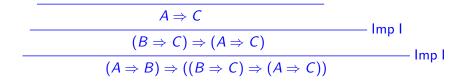
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$$\frac{(B \Rightarrow C) \Rightarrow (A \Rightarrow C)}{(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))} \text{Imp I}$$

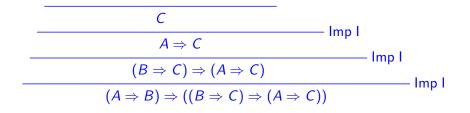
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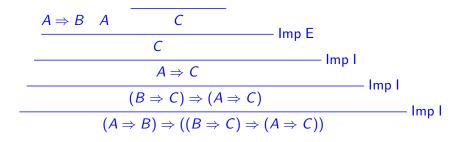
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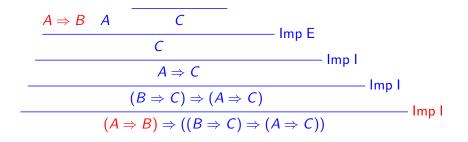
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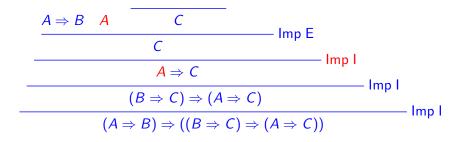
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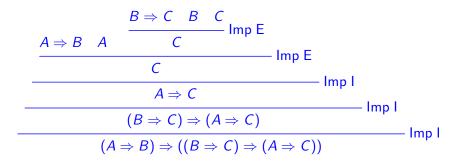


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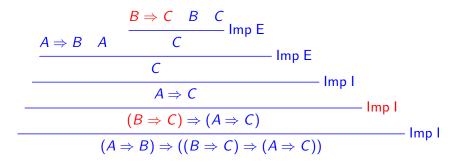


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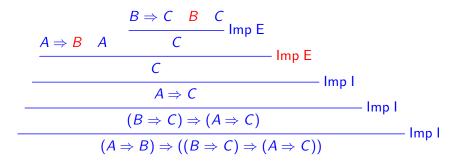
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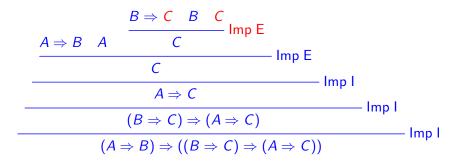
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#### Some Well-Known Derived Rules

#### Modus Ponens

$$\frac{A \Rightarrow B \quad A}{B} \text{ MP}$$

Left Conjunct

$$\frac{A \land B}{A} \text{ AndL}$$

$$\frac{A \Rightarrow B \quad A \quad B}{B} \text{ Imp E}$$

$$\frac{A \wedge B \quad A}{A} \operatorname{And}_{L} \mathsf{E}$$

Right Conjunct

$$\frac{A \land B}{B} \text{ And } \mathbb{R}$$

$$\frac{A \wedge B \quad A}{A} \operatorname{And}_R \mathsf{E}$$

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## Your Turn

#### $(A \land B) \Rightarrow (A \lor B)$

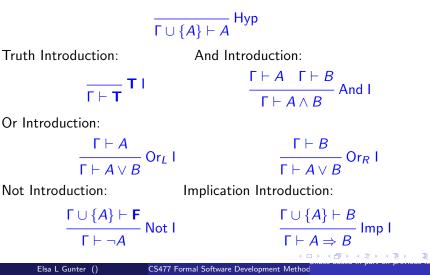
#### Assumptions in Natural Deduction

- Problem: Keeping track of hypotheses and their discharge in Natural Deduction is *HARD*!
- Solution: Use sequents to track hypotheses
- A sequent is a pair of
  - A set of propositions (called assumptions, or hypotheses of sequent) and
  - A proposition (called conclusion of sequent)
- More generally (not here), allow set of hypotheses and set of conclusions

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# Nat. Ded. Introduction Sequent Rules

 $\Gamma$  is set of propositions (assumptions/hypotheses) Hypothesis Introduction:



## Nat. Ded. Elimination Sequent Rules

Γ is set of propositions (assumptions/hypotheses)Not Elimination: Implication Elimination:

 $\frac{\Gamma \vdash \neg A \quad \Gamma \vdash A}{\Gamma \vdash C} \text{ Not E} \qquad \frac{\Gamma \vdash A \Rightarrow B \quad \Gamma \vdash A \quad \Gamma \cup \{B\} \vdash C}{\Gamma \vdash C} \text{ Imp E}$ And Elimination:  $\frac{\Gamma \vdash A \land B \quad \Gamma \cup \{A\} \vdash C}{\Gamma \vdash C} \operatorname{And}_{L} \mathsf{E} \qquad \frac{\Gamma \vdash A \land B \quad \Gamma \cup \{B\} \vdash C}{\Gamma \vdash C} \operatorname{And}_{R} \mathsf{E}$ False Elimination: Or Elimination:  $\frac{\Gamma \vdash \mathbf{F}}{\Gamma \vdash C} \mathbf{F} \mathbf{E} \qquad \frac{\Gamma \vdash A \lor B \quad \Gamma \cup \{A\} \vdash C \quad \Gamma \cup \{B\} \vdash C}{\Gamma \vdash C} \text{ Or } \mathbf{E}$ 

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