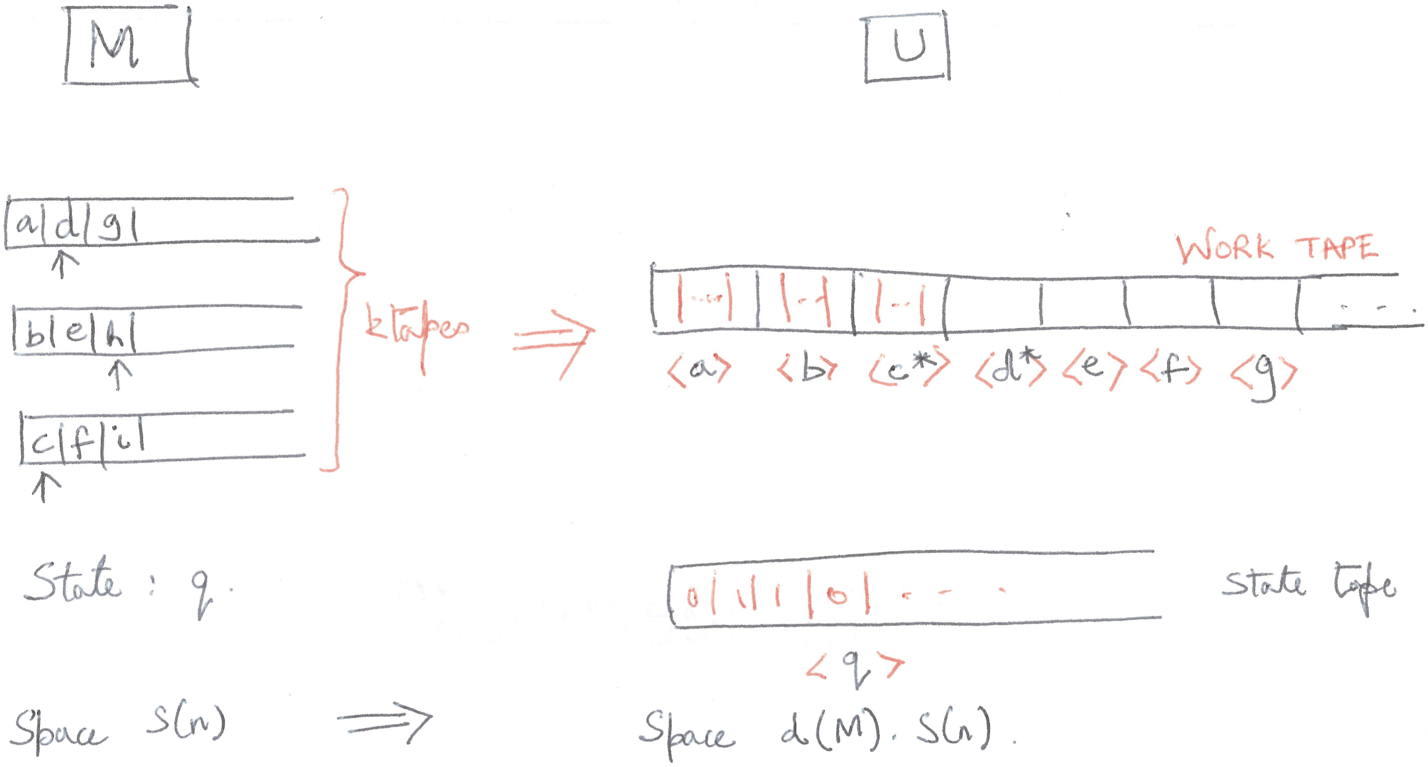
M
U
a d g
b e h
c f i
k tapes
WORK TAPE
⟨a⟩ ⟨b⟩ ⟨c*⟩ ⟨d*⟩ ⟨e⟩ ⟨f⟩ ⟨g⟩
State : q.
0 1 1 0 . . .
State tape
⟨q⟩
Space S(n) ⟹ Space d(M) . S(n) .

Lecture 15: Hierarchy Theorems

Date: October 12, 2023.

Common Complexity Classes:

$\mathrm{L} = \mathrm{DSPACE}(\log n)$	$\mathrm{NL} = \mathrm{NSPACE}(\log n)$
$\mathrm{P} = \mathrm{DTIME}(n^{O(1)})$	$\mathrm{NP} = \mathrm{NTIME}(n^{O(1)})$
$\mathrm{PSPACE} = \mathrm{DSPACE}(n^{O(1)})$	$\mathrm{NPSPACE} = \mathrm{NSPACE}(n^{O(1)})$
$\mathrm{EXP} = \mathrm{DTIME}(2^{n^{O(1)}})$	$\mathrm{NEXP} = \mathrm{NTIME}(2^{n^{O(1)}})$
$\mathrm{EXPSPACE} = \mathrm{DSPACE}(2^{n^{O(1)}})$	$\mathrm{NEXPSPACE} = \mathrm{NSPACE}(2^{n^{O(1)}})$

Relationships between complexity classes thus far

$$\mathrm{L} \subseteq \mathrm{NL} \subseteq \mathrm{P} \subseteq \mathrm{NP} \subseteq \underset{\substack{= \\ \mathrm{NPSPACE}}}{\mathrm{PSPACE}} \subseteq \mathrm{EXP} \subseteq \mathrm{NEXP} \subseteq \underset{\substack{= \\ \mathrm{NEXPSPACE}}}{\mathrm{EXPSPACE}}$$

Little oh: $f(n) = o(g(n))$ if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$. $\forall \epsilon\ \exists n_0\ \forall n \geqslant n_0\ \ g(n) > \epsilon f(n)$.

Time Constructible Functions: A function $T : \mathbb{N} \to \mathbb{N}$ is said to be time constructible if $T(n) \geq n$ and there is a Turing machine M such that M on input x, outputs a string of length $T(|x|)$ in time $T(|x|)$.

Space Constructible Functions: A function $S : \mathbb{N} \to \mathbb{N}$ is said to be space constructible if there is a Turing machine M such that M on input x, outputs a string of length $S(|x|)$ in space $S(|x|)$.

Theorem 1 (Universal Simulation). *For any time constructible function T, there is a Turing machine U such that U runs in time $O(T(n)\log T(n))$ and $\mathbf{L}(U) = \{\langle M, x\rangle \mid M$ accepts x in time $T(|x|)\}$.*

For any space constructible function S, there is a Turing machine U such that U runs in space $O(S(n))$ and $\mathbf{L}(U) = \{\langle M, x\rangle \mid M$ accepts x in space $S(|x|)\}$.

$n, n^k, \log n, 2^n$ – time/space constructible
If f and g are time/space constructible then $f+g$, $f \cdot g$, 2^f, $\log f$, $f(g(n))$ are also time/space constrible.

Theorem 2 (Space Hierarchy). *If S, S' are space constructible functions such that $S(n) = o(S'(n))$ then* $\text{DSPACE}(S(n)) \subsetneq \text{DSPACE}(S'(n))$. $\exists A \in \text{DSPACE}(S'(n)) \setminus \text{DSPACE}(S(n))$

Language A is the one recognized by the following m/c

$A \in \text{DSPACE}(S'(n))$

$A \notin \text{DSPACE}(S(n))$ when $S(n) = o(S'(n))$

Suppose (for contradiction) M_j accepts A in space bound $S(n)$.

- If $j \in L(M_j)$ then $j \notin A$.
- If $j \notin L(M_j)$ then $j \in A$.

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$j \in (L(M_j) \setminus A) \cup (A \setminus L(M_j))$

Input x

Compute $S'(|x|)$ and mark cells.

If x is of the form $0^k 1 w$ then

Run U on $\langle M_{\cancel{x}w}, x \rangle$ in space bound $S'(|x|)$ and time bound $2^{S'(|x|)}$

if M_x ~~rejects~~ doesn't accept x when U uses $S'(|x|)$ cells then accept x.

else

reject x.

$\rightarrow x = \boxed{0^k 1 \langle M \rangle}$ $\hookrightarrow$ as the code of M.

Since $S(n) = o(S'(n))$ that for every w, $d(M_w)$

$\exists n_0 . \forall n \geq n_0 \quad S'(n) > d(M_w) . S(n)$.

Run M_w on $0^{n_0} 1 w$ then the $S'(n)$-bounded simulation of M_w on the universal TM will run to completion.

If $0^{n_0} 1 w \in L(M_w)$ then $0^{n_0} 1 w \notin A$

If $0^{n_0} 1 w \notin L(M_w)$ then $0^{n_0} 1 w \in A$

where n_0 is the number s.t. $\forall n \geq n_0$ $S'(n) > d(M_w) \cdot S(n)$.

Theorem 3 (Time Hierarchy). *If T, T' are time constructible functions such that $T(n) \log T(n) = o(T'(n))$ then* $\text{DTIME}(T(n)) \subsetneq \text{DTIME}(T'(n))$.

Theorem 4. NSPACE(n^k) $\subsetneq$ NSPACE(n^{k+1}), *for any* k.

Claim: If NSPACE(n^k) = NSPACE(n^{k+1}) then NSPACE(n^{k+1}) = NSPACE(n^{k+2})

Proof: NSPACE(n^{k+1}) $\subseteq$ NSPACE(n^{k+2})

Need to show: NSPACE(n^{k+2}) $\subseteq$ NSPACE(n^{k+1})

Let $A \in$ NSPACE(n^{k+2}). $\exists$ NTM M that is n^{k+2}-space bounded and $L(M) = A$. WLOG, $A \subseteq \{0,1\}^*$.

$$A' = \{ x\#^{|x|^{\frac{k+2}{k+1}} - |x|} \mid x \in A \} = \{ x\#^{n^{\frac{k+2}{k+1}} - n} \mid x \in A \text{ and } |x| = n \}$$

Algo A': Input w.

Check $w \in \{0,1\}^* \#^*$. Let $w = x\#^\ell$

Check $\ell = |x|^{k+2/k+1} - |x|$.

$\rightarrow$ Run M on x.

Running ~~time~~ space to solve A' on $x\#^\ell$ $= |x|^{k+2} = |w|^{k+1}$

$$\begin{cases} w = x\#^\ell \\ |w| = |x| + \ell \\ \quad = |x|^{k+2/k+1} \end{cases}$$

$A' \in$ NSPACE(n^{k+1}). $\exists N$ that is n^k-space bounded and $A' = L(N)$.

New Algo for A: Input x.

Create $w = x\#^{|x|^{k+2/k+1} - |x|}$

Run N on w. $\leftarrow$

Space used by new algo for A on x: $(|x|^{k+2/k+1})^k$

$$= |x|^{k(k+2)/k+1} < |x|^{k+1}$$

~~$\frac{k(k+2)}{k+1} \leq k+1$~~

If NSPACE(n^k) = NSPACE(n^{k+1}) Then NSPACE(n^k) = NSPACE(n^{2k})

= NSPACE(n^{3k})

DSPACE(n^k) $\subseteq$ NSPACE(n^k) $\subseteq$ DSPACE(n^{2k}) $\subseteq$ NSPACE(n^{2k})

NSPACE(n^k) $\subseteq$ DSPACE(n^{2k}) $\neq$ DSPACE(n^{3k}) $\subseteq$ NSPACE(n^{3k})

Theorem 5 (Nondeterministic Space Hierarchy Theorem). *If* T, T' *are time constructible functions satisfying* $T(n+1) = o(T(n))$ *then* NTIME($T(n)$) $\subsetneq$ NTIME($T'(n)$).