Lecture 7: Pushdown Automata and Context Free Languages

Date: September 12, 2023.

Proposition 1. *The following languages are not context-free:* $L_{anbncn} = \{a^n b^n c^n \mid n \geq 0\}$ *and* $A = \{ww \mid w \in \{0,1\}^*\}$

Proposition 2. *Context-free languages are not closed under complementation or intersection.*

$\exists$ CFL A, B, C s.t $\bar{A}$, $B \cap C$ are not context free.

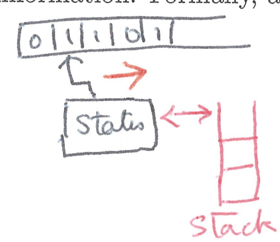
$L = L(((0+1)(0+1))^*(0+1)) \cup \underbrace{\{uv \mid |u|=|v| \text{ and } u \neq v\}}_{\text{is CFL (Homework 2)}}$. is CFL

$\bar{L} = A$ is not CFL.

$B = \{a^i b^i c^j \mid i,j \geq 0\}$ is CFL (exercise) $\quad C = \{a^i b^j c^j \mid i,j \geq 0\}$ is CFL

$B \cap C = L_{anbncn}$ is not CFL.

(Nondeterministic) Pushdown Automaton (PDA): Like an NFA with ϵ-transitions, except it has an **stack/pushdown store** in which it can record an unbounded amount of information. Formally, a PDA is $M = (Q, \Sigma, \Gamma, \delta, s, \perp, F)$ where



- Q is the (finite) set of states,
- Σ is the input alphabet,
- Γ is the stack alphabet,
- $\delta : (Q \times (\Sigma \cup \{\epsilon\}) \times \Gamma) \times (Q \times \Gamma^*)$ is the *transition relation*,
- $s \in Q$ is the start state,
- $\perp \in \Gamma$ is the initial stack symbol,
- $F \subseteq Q$ is the set of final states.

(a) Read an input symbol
(b) Pop the top the stack.
(c) Based on the current state, it will push a bunch symbol. and change state.

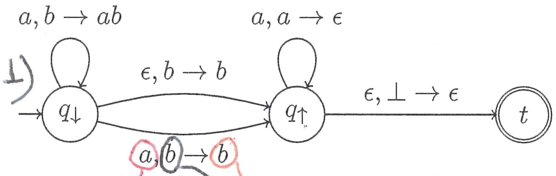
If $((p, a, A), (q, B_1 B_2 \cdots B_k)) \in \delta$ then it means that when M is in state p, reading symbol $a \in \Sigma$ (or choosing not to read any input symbol if $a = \epsilon$), with A on top of its stack, it could choose to pop A from the stack, change state to q, and push $B_1 B_2 \cdots B_k$ on the stack (with B_k being pushed first and B_1 last).

A **configuration** of M is an element $(q, w, \alpha) \in Q \times \Sigma^* \times \Gamma^*$ describing the current state (q), the input remaining to be processed (w with the leftmost symbol being the next one to be read), and current stack contents (α with the leftmost symbol being the top). The **initial/start configuration** on input x is $(s, x, \perp)$. The **next configuration relation**, $\xrightarrow[M]{1}$, which describes a step of M, is defined as follows.

- If $((p, a, A), (q, \gamma)) \in \delta$ $(a \in \Sigma)$ then for any $y \in \Sigma^*$ and $\beta \in \Gamma^*$, $(p, ay, A\beta) \xrightarrow[M]{1} (q, y, \gamma\beta)$.
- If $((p, \epsilon, A), (q, \gamma)) \in \delta$ $\cancel{(a \in \Sigma)}$ then for any $y \in \Sigma^*$ and $\beta \in \Gamma^*$, $(p, y, A\beta) \xrightarrow[M]{1} (q, y, \gamma\beta)$.

Taking $\xrightarrow[M]{*}$ to be reflexive, transitive closure of $\xrightarrow[M]{1}$, we can define $\mathbf{L}(M) = \{x \in \Sigma^* \mid \exists \gamma \in \Gamma^*,\ q \in F.\ (s, x, \perp) \xrightarrow[M]{*} (q, \epsilon, \gamma)\}$.

Problem 1. Consider the PDA $M = (\{q_\downarrow, q_\uparrow, t\}, \{0,1\}, \{\bot, 0, 1\}, \delta, q_\downarrow, \bot, \{t\})$. Describe computations of M on 1100, 101 and 1001.



$a \in \{0,1\}$, $b \in \{\bot, 0, 1\}$

$(q_\downarrow, 1100, \bot) \to (q_\downarrow, 100, 1\bot) \to \cdots \to (q_\downarrow, \epsilon, 0011\bot)$

$\to (q_\uparrow, 100, \bot) \to (t, 100, \epsilon)$

$(q_\downarrow, 101, \bot) \xrightarrow[M]{1} (q_\downarrow, 01, 1\bot) \xrightarrow[M]{1} (q_\uparrow, 1, 1\bot)$

$\xrightarrow[M]{1} (q_\uparrow, \epsilon, \bot) \xrightarrow[M]{1} (t, \epsilon, \epsilon)$ ✓

Palindrome : $w = \text{reverse}(w)$

$L(M) = \{ w \in \{0,1\}^* \mid w \text{ is a palindrome} \}$

Theorem 3. *If L is a context-free language then there is a PDA M such that $\mathbf{L}(M) = L$.*

$\exists\, G = (N, \Sigma, P, S)$ s.t $L(G) = L$.

Goal : Design an algorithm (PDA) s.t it check if the input $w \in L(G)$

or check if $S \xrightarrow[G]{*} w$.

M : Will guess a leftmost derivation of G that produces w.

Leftmost derivation : is a derivation where the leftmost non-terminal is replaced in every step.

$\in N$

String in derivation : $x A \alpha$, $x \in \Sigma^*$

Example : $S \to AC$

$A \to \epsilon \mid aAb$

$C \to \epsilon \mid cC$

$S \xrightarrow[G]{1} AC \xrightarrow[G]{1} aA\boxed{bC} \xrightarrow[G]{1} abC$

$\xrightarrow[G]{1} abcC \xrightarrow[G]{1} abc$

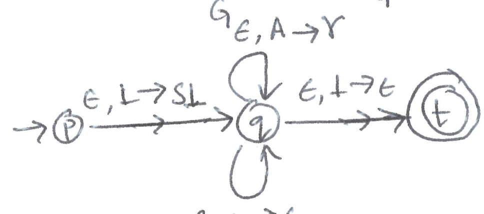
$M = (\{p, q, t\}, \Sigma, N \cup \Sigma \cup \{\bot\}, \delta, p, \bot, \{t\})$

Transitions : $((p, \epsilon, \bot), (q, S\bot))$

$((q, \epsilon, A), (q, \gamma))$ where $A \to \gamma \in P$

$((q, a, a), (q, \epsilon))$ where $a \in \Sigma$

$((q, \epsilon, \bot), (t, \epsilon))$



$(p, abc, \bot) \xrightarrow[M]{1} (q, abc, S\bot)$

$\xrightarrow[M]{1} (q, abc, AC\bot) \xrightarrow[M]{1} (q, abc, aAbC\bot)$

$\xrightarrow[M]{1} (q, bc, AbC\bot) \xrightarrow[M]{1} (q, bc, bC\bot)$