# Network Flow Algorithms

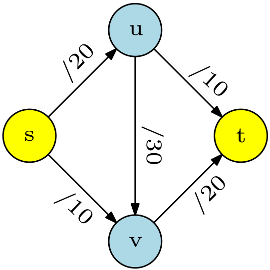
Lecture 17

March 27, 2013

# Part I

# Algorithm(s) for Maximum Flow

# Greedy Approach

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2. Find a $\mathbf{s}$-$\mathbf{t}$ path $\mathbf{P}$ with $\mathbf{f(e) < c(e)}$ for every edge $\mathbf{e} \in \mathbf{P}$.
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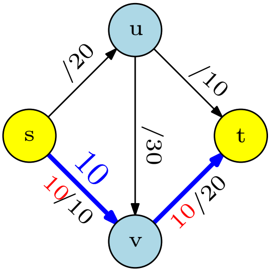


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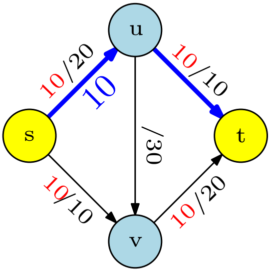


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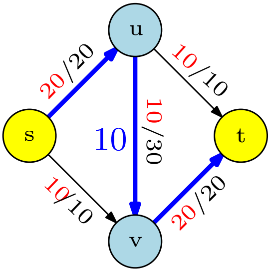


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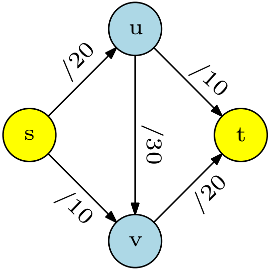


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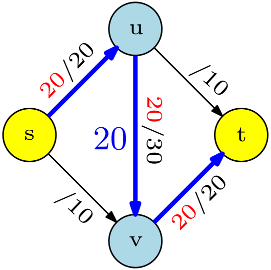


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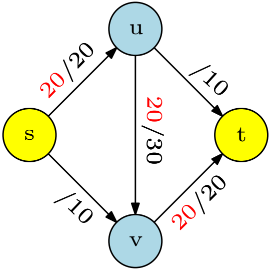


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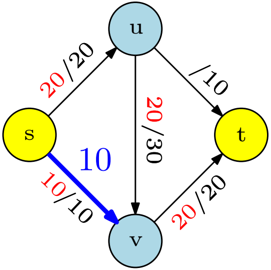


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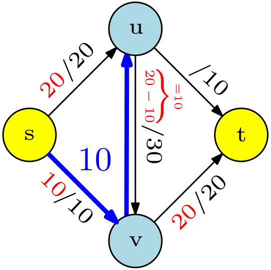


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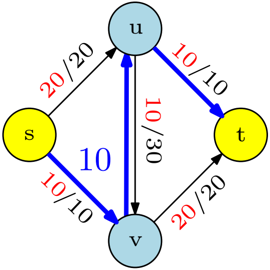


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# Residual Graph

## The "leftover" graph

### Definition

For a network $G = (V, E)$ and flow $f$, the **residual graph** $G_f = (V', E')$ of $G$ with respect to $f$ is

1. $V' = V$,
2. **Forward Edges**: For each edge $e \in E$ with $f(e) < c(e)$, we add $e \in E'$ with capacity $c(e) - f(e)$.
3. **Backward Edges**: For each edge $e = (u, v) \in E$ with $f(e) > 0$, we add $(v, u) \in E'$ with capacity $f(e)$.

# Residual Graph Example

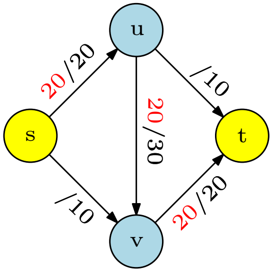


Figure: Flow on edges is indicated in red

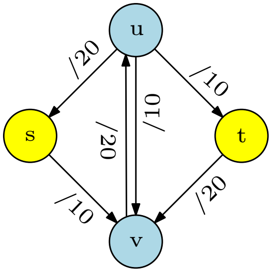


Figure: Residual Graph

# Residual Graph Property

**Observation:** Residual graph captures the "residual" problem exactly.

### Lemma

*Let $\mathbf{f}$ be a flow in $\mathbf{G}$ and $\mathbf{G_f}$ be the residual graph. If $\mathbf{f'}$ is a flow in $\mathbf{G_f}$ then $\mathbf{f} + \mathbf{f'}$ is a flow in $\mathbf{G}$ of value $\mathbf{v(f)} + \mathbf{v(f')}$.*

### Lemma

*Let $\mathbf{f}$ and $\mathbf{f'}$ be two flows in $\mathbf{G}$ with $\mathbf{v(f')} \geq \mathbf{v(f)}$. Then there is a flow $\mathbf{f''}$ of value $\mathbf{v(f')} - \mathbf{v(f)}$ in $\mathbf{G_f}$.*

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# Residual Graph Property: Implication

*Recursive* algorithm for finding a maximum flow:

**MaxFlow**($G, s, t$):
    **if** the flow from $s$ to $t$ is $0$ **then**
        **return** $0$
    Find any flow $f$ with $v(f) > 0$ in $G$
    Recursively compute a maximum flow $f'$ in $G_f$
    Output the flow $f + f'$

*Iterative* algorithm for finding a maximum flow:

**MaxFlow**($G, s, t$):
    Start with flow $f$ that is $0$ on all edges
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    G_f is residual graph of G with respect to f
    while G_f has a simple s-t path do
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augment(f,P)
    let b be bottleneck capacity,
        i.e., min capacity of edges in P (in G_f)
    for each edge (u,v) in P do
        if e = (u,v) is a forward edge then
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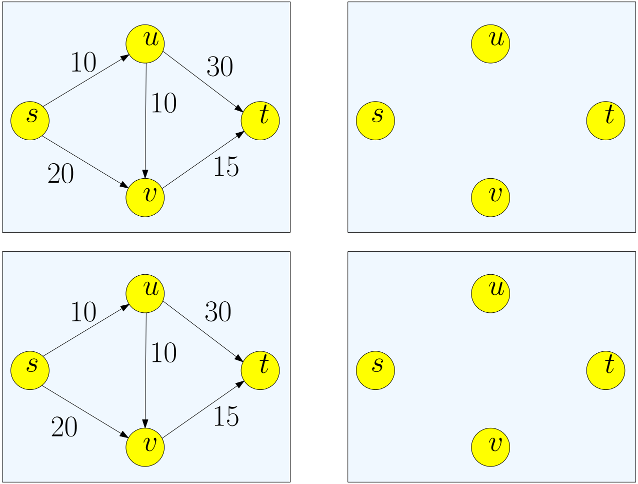
# Example



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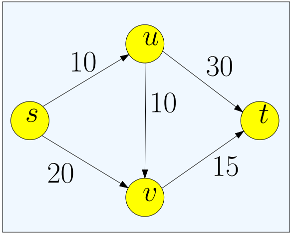


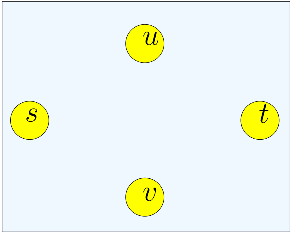


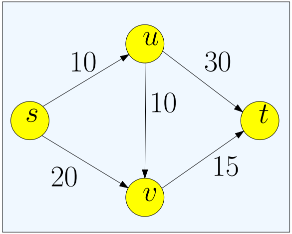


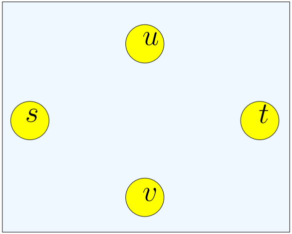

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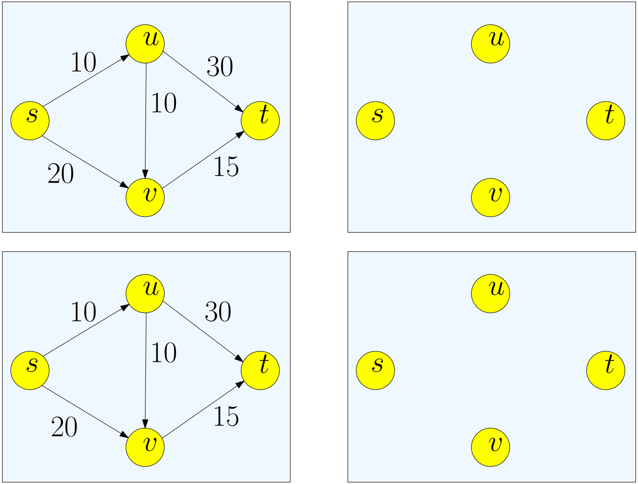

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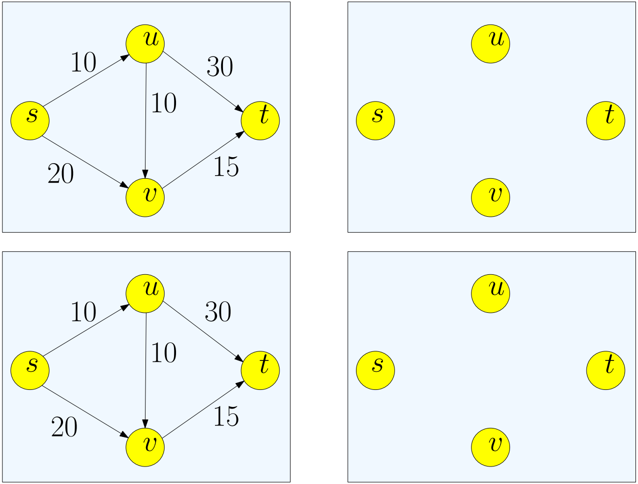

# Properties about Augmentation: Flow

## Lemma

*If* $\mathbf{f}$ *is a flow and* $\mathbf{P}$ *is a simple* $\mathbf{s}$-$\mathbf{t}$ *path in* $\mathbf{G_f}$*, then* $\mathbf{f}' = \mathtt{augment}(\mathbf{f}, \mathbf{P})$ *is also a flow.*

## Proof.

Verify that $\mathbf{f}'$ is a flow. Let $\mathbf{b}$ be augmentation amount.

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# Properties of Augmentation

## Conservation Constraint

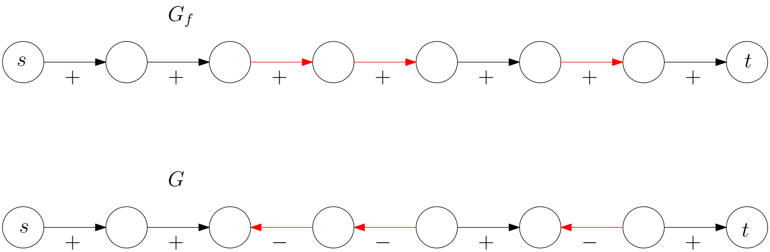


Figure: Augmenting path $\mathbf{P}$ in $\mathbf{G_f}$ and corresponding change of flow in $\mathbf{G}$. Red edges are backward edges.

# Properties of Augmentation

## Integer Flow

### Lemma

*At every stage of the Ford-Fulkerson algorithm, the flow values on the edges (i.e.,* $\mathbf{f(e)}$*, for all edges* $\mathbf{e}$*) and the residual capacities in* $\mathbf{G_f}$ *are integers.*

### Proof.

Initial flow and residual capacities are integers. Suppose lemma holds for $\mathbf{j}$ iterations. Then in $\mathbf{(j+1)}$st iteration, minimum capacity edge $\mathbf{b}$ is an integer, and so flow after augmentation is an integer. □

# Progress in Ford-Fulkerson

## Proposition

*Let* $\mathbf{f}$ *be a flow and* $\mathbf{f}'$ *be flow after one augmentation. Then* $\mathbf{v(f)} < \mathbf{v(f')}$.

## Proof.

Let $\mathbf{P}$ be an augmenting path, i.e., $\mathbf{P}$ is a simple $\mathbf{s}$-$\mathbf{t}$ path in residual graph. We have the following.

1. First edge $\mathbf{e}$ in $\mathbf{P}$ must leave $\mathbf{s}$.
2. Original network $\mathbf{G}$ has no incoming edges to $\mathbf{s}$; hence $\mathbf{e}$ is a forward edge.
3. $\mathbf{P}$ is simple and so never returns to $\mathbf{s}$.
4. Thus, value of flow increases by the flow on edge $\mathbf{e}$. □

# Termination proof for integral flow

## Theorem

*Let* $\mathbf{C}$ *be the minimum cut value; in particular* $\mathbf{C} \leq \sum_{\mathbf{e}\ \textit{out of}\ \mathbf{s}} \mathbf{c}(\mathbf{e})$. *Ford-Fulkerson algorithm terminates after finding at most* $\mathbf{C}$ *augmenting paths.*

## Proof.

The value of the flow increases by at least $\mathbf{1}$ after each augmentation. Maximum value of flow is at most $\mathbf{C}$. □

## Running time

1. Number of iterations $\leq \mathbf{C}$.
2. Number of edges in $\mathbf{G_f} \leq \mathbf{2m}$.
3. Time to find augmenting path is $\mathbf{O(n+m)}$.
4. Running time is $\mathbf{O(C(n+m))}$ (or $\mathbf{O(mC)}$).

Running time $= \mathbf{O(mC)}$ is not polynomial. Can the running time be as $\mathbf{\Omega(mC)}$ or is our analysis weak?




Ford-Fulkerson can take $\mathbf{\Omega(C)}$ iterations.

# Efficiency of Ford-Fulkerson

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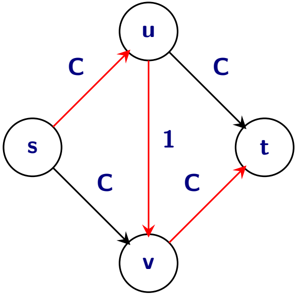





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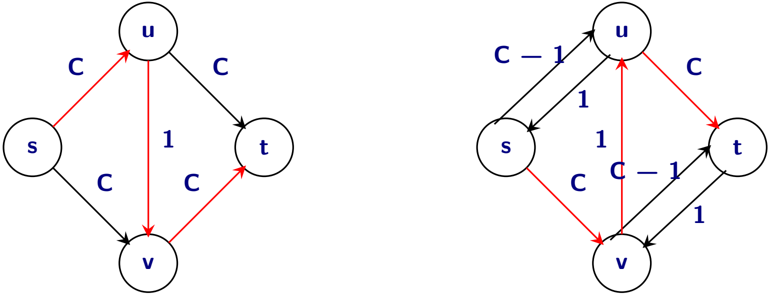


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# Correctness of Ford-Fulkerson

## Why the augmenting path approach works

Question: When the algorithm terminates, is the flow computed the maximum **s**-**t** flow?

Proof idea: show a cut of value equal to the flow. Also shows that maximum flow is equal to minimum cut!

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# Recalling Cuts

## Definition

Given a flow network an **s-t cut** is a set of edges $\mathbf{E'} \subset \mathbf{E}$ such that removing $\mathbf{E'}$ *disconnects* $\mathbf{s}$ from $\mathbf{t}$: in other words there is no directed $\mathbf{s} \to \mathbf{t}$ path in $\mathbf{E} - \mathbf{E'}$. **Capacity** of cut $\mathbf{E'}$ is $\sum_{\mathbf{e} \in \mathbf{E'}} \mathbf{c(e)}$.

Let $\mathbf{A} \subset \mathbf{V}$ such that

1. $\mathbf{s} \in \mathbf{A}$, $\mathbf{t} \notin \mathbf{A}$, and
2. $\mathbf{B} = \mathbf{V} \setminus -\mathbf{A}$ and hence $\mathbf{t} \in \mathbf{B}$.

Define $\mathbf{(A, B)} = \{\mathbf{(u, v)} \in \mathbf{E} \mid \mathbf{u} \in \mathbf{A}, \mathbf{v} \in \mathbf{B}\}$

## Claim

$\mathbf{(A, B)}$ *is an* $\mathbf{s}$-$\mathbf{t}$ *cut.*

Recall: Every *minimal* $\mathbf{s}$-$\mathbf{t}$ cut $\mathbf{E'}$ is a cut of the form $\mathbf{(A, B)}$.

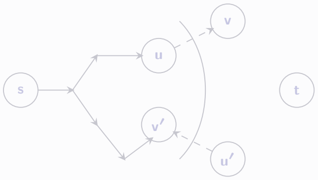
# Ford-Fulkerson Correctness

## Lemma

*If there is no* $\mathbf{s}$-$\mathbf{t}$ *path in* $\mathbf{G_f}$ *then there is some cut* $\mathbf{(A, B)}$ *such that* $\mathbf{v(f) = c(A, B)}$

## Proof.

Let $\mathbf{A}$ be all vertices reachable from $\mathbf{s}$ in $\mathbf{G_f}$; $\mathbf{B = V \setminus A}$.




1. $\mathbf{s \in A}$ and $\mathbf{t \in B}$. So $\mathbf{(A, B)}$ is an $\mathbf{s}$-$\mathbf{t}$ cut in $\mathbf{G}$.
2. If $\mathbf{e = (u, v) \in G}$ with $\mathbf{u \in A}$ and $\mathbf{v \in B}$, then $\mathbf{f(e) = c(e)}$ (saturated edge) because otherwise $\mathbf{v}$ is reachable from $\mathbf{s}$ in $\mathbf{G_f}$.

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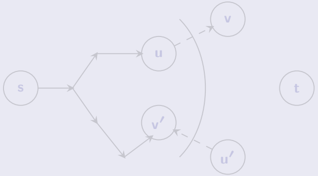


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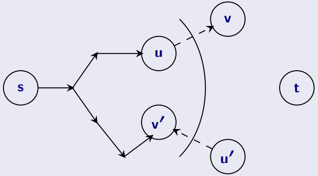


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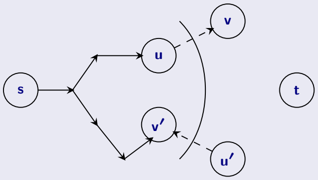


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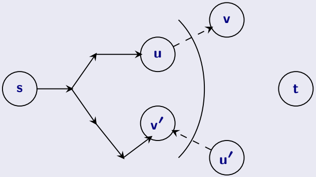
□

# Lemma Proof Continued

## Proof.

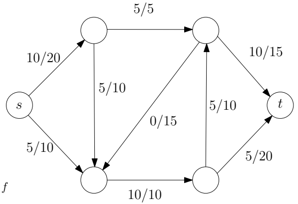



1. If $\mathbf{e} = (\mathbf{u}', \mathbf{v}') \in \mathbf{G}$ with $\mathbf{u}' \in \mathbf{B}$ and $\mathbf{v}' \in \mathbf{A}$, then $\mathbf{f(e) = 0}$ because otherwise $\mathbf{u}'$ is reachable from $\mathbf{s}$ in $\mathbf{G_f}$
2. Thus,

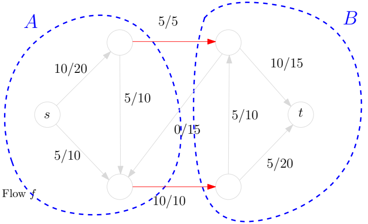
$$
\begin{aligned}
\mathbf{v(f)} &= \mathbf{f}^{\text{out}}(\mathbf{A}) - \mathbf{f}^{\text{in}}(\mathbf{A}) \\
&= \mathbf{f}^{\text{out}}(\mathbf{A}) - \mathbf{0} \\
&= \mathbf{c(A, B)} - \mathbf{0} \\
&= \mathbf{c(A, B)}.
\end{aligned}
$$

$\square$

# Example




Flow $f$

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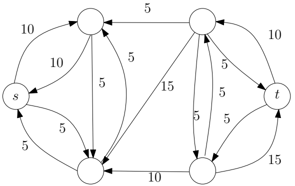
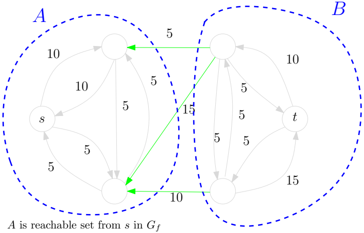



Residual graph $G_f$: no $s$-$t$ path

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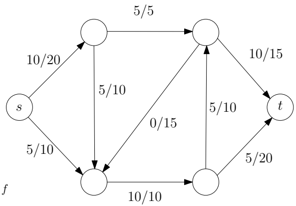


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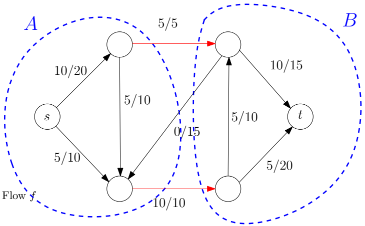


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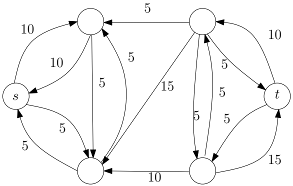


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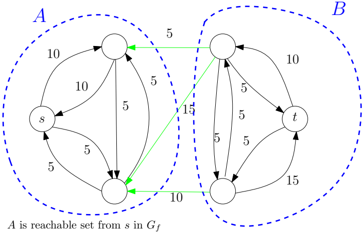


$A$ is reachable set from $s$ in $G_f$

# Ford-Fulkerson Correctness

## Theorem

*The flow returned by the algorithm is the maximum flow.*

## Proof.

1. For any flow $\mathbf{f}$ and $\mathbf{s}$-$\mathbf{t}$ cut $(\mathbf{A}, \mathbf{B})$, $\mathbf{v(f)} \leq \mathbf{c(A, B)}$.
2. For flow $\mathbf{f}^*$ returned by algorithm, $\mathbf{v(f^*)} = \mathbf{c(A^*, B^*)}$ for some $\mathbf{s}$-$\mathbf{t}$ cut $(\mathbf{A}^*, \mathbf{B}^*)$.
3. Hence, $\mathbf{f}^*$ is maximum.

□

# Max-Flow Min-Cut Theorem and Integrality of Flows

## Theorem

*For any network* $\mathbf{G}$*, the value of a maximum* $\mathbf{s}$-$\mathbf{t}$ *flow is equal to the capacity of the minimum* $\mathbf{s}$-$\mathbf{t}$ *cut.*

## Proof.

Ford-Fulkerson algorithm terminates with a maximum flow of value equal to the capacity of a (minimum) cut. □

# Max-Flow Min-Cut Theorem and Integrality of Flows

## Theorem

*For any network* **G** *with integer capacities, there is a maximum* **s-t** *flow that is integer valued.*

## Proof.

Ford-Fulkerson algorithm produces an integer valued flow when capacities are integers. □

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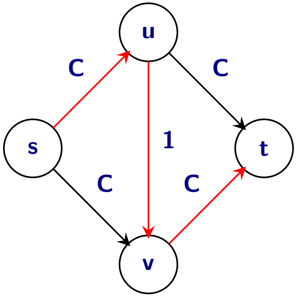

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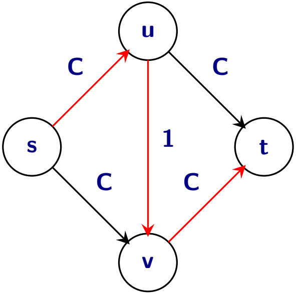


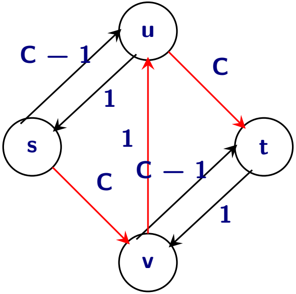

# Polynomial Time Algorithms

Question: Is there a polynomial time algorithm for maxflow?

Question: Is there a variant of Ford-Fulkerson that leads to a polynomial time algorithm? Can we choose an augmenting path in some clever way? Yes! Two variants.

1. Choose the augmenting path with largest bottleneck capacity.
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# Augmenting Paths with Large Bottleneck Capacity

1. Pick augmenting paths with largest bottleneck capacity in each iteration of Ford-Fulkerson.
2. How do we find path with largest bottleneck capacity?
    1. Assume we know $\Delta$ the bottleneck capacity
    2. Remove all edges with residual capacity $\leq \Delta$
    3. Check if there is a path from $s$ to $t$
    4. Do binary search to find largest $\Delta$
    5. Running time: $O(m \log C)$
3. Can we bound the number of augmentations? Can show that in $O(m \log C)$ augmentations the algorithm reaches a max flow. This leads to an $O(m^2 \log^2 C)$ time algorithm.

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# Augmenting Paths with Large Bottleneck Capacity

How do we find path with largest bottleneck capacity?

1. Max bottleneck capacity is one of the edge capacities. Why?
2. Can do binary search on the edge capacities. First, sort the edges by their capacities and then do binary search on that array as before.
3. Algorithm's running time is $\mathbf{O(m \log m)}$.
4. Different algorithm that also leads to $\mathbf{O(m \log m)}$ time algorithm by adapting Prim's algorithm.

# Removing Dependence on $C$

1. **Dinic [1970]**, **Edmonds and Karp [1972]**
   Picking augmenting paths with fewest number of edges yields a $O(m^2 n)$ algorithm, i.e., independent of $C$. Such an algorithm is called a strongly polynomial time algorithm since the running time does not depend on the numbers (assuming RAM model). (Many implementation of Ford-Fulkerson would actually use shortest augmenting path if they use **BFS** to find an $s$-$t$ path).
2. Further improvements can yield algorithms running in $O(mn \log n)$, or $O(n^3)$.

# Ford-Fulkerson Algorithm

```
algEdmondsKarp
    for every edge e, f(e) = 0
    G_f is residual graph of G with respect to f
    while G_f has a simple s-t path do
        Perform BFS in G_f
        P:  shortest s-t path in G_f
        f = augment(f, P)
        Construct new residual graph G_f.
```

Running time $O(m^2 n)$.

# Finding a Minimum Cut

Question: How do we find an actual minimum $\mathbf{s}$-$\mathbf{t}$ cut?

Proof gives the algorithm!

1. Compute an $\mathbf{s}$-$\mathbf{t}$ maximum flow $\mathbf{f}$ in $\mathbf{G}$
2. Obtain the residual graph $\mathbf{G_f}$
3. Find the nodes $\mathbf{A}$ reachable from $\mathbf{s}$ in $\mathbf{G_f}$
4. Output the cut $\mathbf{(A, B) = \{(u, v) \mid u \in A, v \in B\}}$. Note: The cut is found in $\mathbf{G}$ while $\mathbf{A}$ is found in $\mathbf{G_f}$

Running time is essentially the same as finding a maximum flow.

Note: Given $\mathbf{G}$ and a flow $\mathbf{f}$ there is a linear time algorithm to check if $\mathbf{f}$ is a maximum flow and if it is, outputs a minimum cut. How?

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Dinic, E. A. (1970). Algorithm for solution of a problem of maximum flow in a network with power estimation. *Soviet Math. Doklady*, 11:1277–1280.

Edmonds, J. and Karp, R. M. (1972). Theoretical improvements in algorithmic efficiency for network flow problems. *J. Assoc. Comput. Mach.*, 19(2):248–264.