

Randomized Algorithms: QuickSort and QuickSelect

Lecture 14

March 8, 2013

Part I

Slick analysis of QuickSort

A Slick Analysis of QuickSort

Let $Q(A)$ be number of comparisons done on input array A :

- 1 For $1 \leq i < j \leq n$ let R_{ij} be the event that rank i element is compared with rank j element.
- 2 X_{ij} is the indicator random variable for R_{ij} . That is, $X_{ij} = 1$ if rank i is compared with rank j element, otherwise 0 .

$$Q(A) = \sum_{1 \leq i < j \leq n} X_{ij}$$

and hence by linearity of expectation,

$$E[Q(A)] = \sum_{1 \leq i < j \leq n} E[X_{ij}] = \sum_{1 \leq i < j \leq n} \Pr[R_{ij}].$$

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A Slick Analysis of QuickSort

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Question: What is $\Pr[R_{ij}]$?

7 5 9 1 3 4 8 6

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With ranks: 6 4 8 1 2 3 7 5

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With ranks: 6 4 8 1 2 3 7 5

As such, probability of comparing 5 to 8 is $\Pr[R_{4,7}]$.

A Slick Analysis of QuickSort

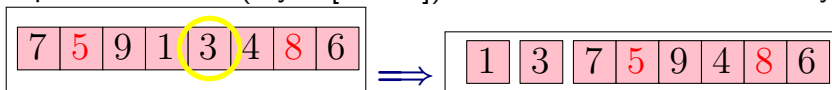
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With ranks: 6 4 8 1 2 3 7 5

- ① If pivot too small (say **3** [rank 2]). Partition and call recursively:



Decision if to compare **5** to **8** is moved to subproblem.

A Slick Analysis of QuickSort

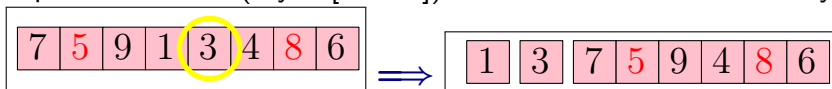
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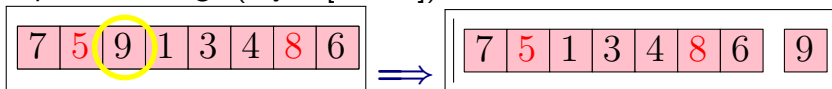
With ranks: 6 4 8 1 2 3 7 5

- ① If pivot too small (say **3** [rank 2]). Partition and call recursively:



Decision if to compare **5** to **8** is moved to subproblem.

- ② If pivot too large (say **9** [rank 8]):



Decision if to compare **5** to **8** moved to subproblem.

A Slick Analysis of QuickSort

Question: What is $\Pr[R_{i,j}]$?

7	5	9	1	3	4	8	6
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As such, probability of comparing **5** to **8** is $\Pr[R_{4,7}]$.

① If pivot is **5** (rank 4). Bingo!

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1	3	4	5	7	9	8	6
---	---	---	---	---	---	---	---

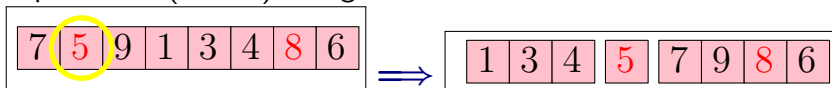
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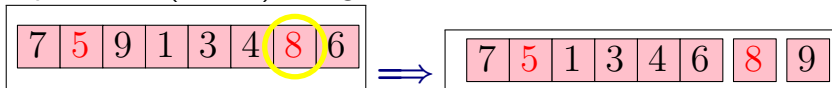
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As such, probability of comparing **5** to **8** is $\Pr[R_{4,7}]$.

- ① If pivot is **5** (rank 4). Bingo!



- ② If pivot is **8** (rank 7). Bingo!



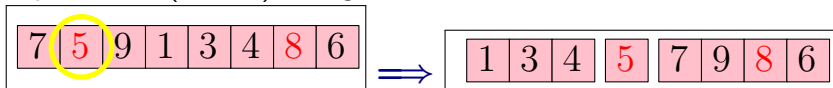
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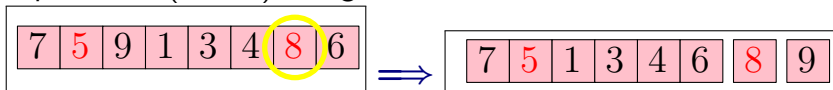
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As such, probability of comparing **5** to **8** is $\Pr[R_{4,7}]$.

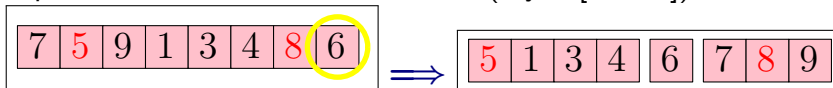
- ① If pivot is **5** (rank 4). Bingo!



- ② If pivot is **8** (rank 7). Bingo!



- ③ If pivot is in between the two numbers (say **6** [rank 5]):



5 and **8** will never be compared to each other.

A Slick Analysis of QuickSort

Question: What is $\Pr[R_{i,j}]$?

Conclusion:

$R_{i,j}$ happens if and only if:

i th or j th ranked element is the first pivot out of
 i th to j th ranked elements.

How to analyze this?

Thinking acrobatics!

- 1 Assign every element in the array a random priority (say in $[0, 1]$).
- 2 Choose pivot to be the element with lowest priority in subproblem.
- 3 Equivalent to picking pivot uniformly at random (as **QuickSort** do).

A Slick Analysis of QuickSort

Question: What is $\Pr[R_{i,j}]$?

How to analyze this?

Thinking acrobatics!

- 1 Assign every element in the array a random priority (say in $[0, 1]$).
- 2 Choose pivot to be the element with lowest priority in subproblem.

$\Rightarrow R_{i,j}$ happens if either i or j have lowest priority out of elements rank i to j ,

There are $k = j - i + 1$ relevant elements.

$$\Pr[R_{i,j}] = \frac{2}{k} = \frac{2}{j - i + 1}.$$

A Slick Analysis of QuickSort

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A Slick Analysis of QuickSort

Question: What is $\Pr[R_{ij}]$?

Lemma

$$\Pr[R_{ij}] = \frac{2}{j-i+1}.$$

Proof.

Let $a_1, \dots, a_i, \dots, a_j, \dots, a_n$ be elements of A in sorted order. Let $S = \{a_i, a_{i+1}, \dots, a_j\}$

Observation: If pivot is chosen outside S then all of S either in left array or right array.

Observation: a_i and a_j separated when a pivot is chosen from S for the first time. Once separated no comparison.

Observation: a_i is compared with a_j if and only if either a_i or a_j is chosen as a pivot from S at separation... □

A Slick Analysis of QuickSort

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A Slick Analysis of QuickSort

Continued...

Lemma

$$\Pr[R_{ij}] = \frac{2}{j-i+1}.$$

Proof.

Let $\mathbf{a}_1, \dots, \mathbf{a}_i, \dots, \mathbf{a}_j, \dots, \mathbf{a}_n$ be sort of \mathbf{A} . Let

$$\mathbf{S} = \{\mathbf{a}_i, \mathbf{a}_{i+1}, \dots, \mathbf{a}_j\}$$

Observation: \mathbf{a}_i is compared with \mathbf{a}_j if and only if either \mathbf{a}_i or \mathbf{a}_j is chosen as a pivot from \mathbf{S} at separation.

Observation: Given that pivot is chosen from \mathbf{S} the probability that it is \mathbf{a}_i or \mathbf{a}_j is exactly $2/|\mathbf{S}| = 2/(j-i+1)$ since the pivot is chosen uniformly at random from the array. □

A Slick Analysis of QuickSort

Continued...

$$E[Q(A)] = \sum_{1 \leq i < j \leq n} E[X_{ij}] = \sum_{1 \leq i < j \leq n} \Pr[R_{ij}].$$

Lemma

$$\Pr[R_{ij}] = \frac{2}{j-i+1}.$$

$$E[Q(A)] = \sum_{1 \leq i < j \leq n} \frac{2}{j-i+1}$$

A Slick Analysis of QuickSort

Continued...

Lemma

$$\Pr[R_{ij}] = \frac{2}{j-i+1}.$$

$$\mathbb{E}[Q(A)] = \sum_{1 \leq i < j \leq n} \Pr[R_{ij}] = \sum_{1 \leq i < j \leq n} \frac{2}{j-i+1}$$

A Slick Analysis of QuickSort

Continued...

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A Slick Analysis of QuickSort

Continued...

Lemma

$$\Pr[R_{ij}] = \frac{2}{j-i+1}.$$

$$\begin{aligned} E[Q(A)] &= \sum_{1 \leq i < j \leq n} \frac{2}{j-i+1} \\ &= \sum_{i=1}^{n-1} \sum_{j=i+1}^n \frac{2}{j-i+1} \end{aligned}$$

A Slick Analysis of QuickSort

Continued...

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$$\Pr[R_{ij}] = \frac{2}{j-i+1}.$$

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A Slick Analysis of QuickSort

Continued...

Lemma

$$\Pr[R_{ij}] = \frac{2}{j-i+1}.$$

$$\mathbb{E}[Q(A)] = 2 \sum_{i=1}^{n-1} \sum_{i < j}^n \frac{1}{j-i+1}$$

A Slick Analysis of QuickSort

Continued...

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$$\Pr[R_{ij}] = \frac{2}{j-i+1}.$$

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A Slick Analysis of QuickSort

Continued...

Lemma

$$\Pr[R_{ij}] = \frac{2}{j-i+1}.$$

$$E[Q(A)] = 2 \sum_{i=1}^{n-1} \sum_{i < j}^n \frac{1}{j-i+1} \leq 2 \sum_{i=1}^{n-1} \sum_{\Delta=2}^{n-i+1} \frac{1}{\Delta}$$

A Slick Analysis of QuickSort

Continued...

Lemma

$$\Pr[R_{ij}] = \frac{2}{j-i+1}.$$

$$\begin{aligned} E[Q(A)] &= 2 \sum_{i=1}^{n-1} \sum_{i < j}^n \frac{1}{j-i+1} \leq 2 \sum_{i=1}^{n-1} \sum_{\Delta=2}^{n-i+1} \frac{1}{\Delta} \\ &\leq 2 \sum_{i=1}^{n-1} (H_{n-i+1} - 1) \leq 2 \sum_{1 \leq i < n} H_n \end{aligned}$$

A Slick Analysis of QuickSort

Continued...

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$$\begin{aligned} E[Q(A)] &= 2 \sum_{i=1}^{n-1} \sum_{i < j}^n \frac{1}{j-i+1} \leq 2 \sum_{i=1}^{n-1} \sum_{\Delta=2}^{n-i+1} \frac{1}{\Delta} \\ &\leq 2 \sum_{i=1}^{n-1} (H_{n-i+1} - 1) \leq 2 \sum_{1 \leq i < n} H_n \\ &\leq 2nH_n = O(n \log n) \end{aligned}$$

Yet another analysis of QuickSort

You should never trust a man who has only one way to spell a word

Consider element **e** in the array.

Consider the subproblems it participates in during **QuickSort** execution:

S₁, **S**₂, ..., **S**_k.

Definition

e is lucky in the **j**th iteration if $|\mathbf{S}_j| \leq (3/4) |\mathbf{S}_{j-1}|$.

Key observation

The event **e** is lucky in **j**th iteration
is independent of
the event that **e** is lucky in **k**th iteration,
(If **j** ≠ **k**)

X_j = 1 iff **e** is lucky in the **j**th iteration.

Yet another analysis of QuickSort

Continued...

Claim

$$\Pr[X_j = 1] = 1/2.$$

Proof.

- ① X_j determined by j recursive subproblem.
- ② Subproblem has $n_{j-1} = |X_{j-1}|$ elements.
- ③ If j th pivot rank $\in [n_{j-1}/4, (3/4)n_{j-1}]$, then e lucky in j th iter.
- ④ Prob. e is lucky $\geq |[n_{j-1}/4, (3/4)n_{j-1}]| / n_{j-1} = 1/2.$ \square

Observation

If $X_1 + X_2 + \dots + X_k = \lceil \log_{4/3} n \rceil$ then e subproblem is of size one.
Done!

Yet another analysis of QuickSort

Continued...

Observation

Probability **e** participates in $\geq k = 40 \lceil \log_{4/3} n \rceil$ subproblems. Is equal to

$$\begin{aligned} \Pr[X_1 + X_2 + \dots + X_k \leq \lceil \log_{4/3} n \rceil] \\ \leq \Pr[X_1 + X_2 + \dots + X_k \leq k/4] \\ \leq 2 \cdot 0.68^{k/4} \leq 1/n^5. \end{aligned}$$

Conclusion

QuickSort takes $O(n \log n)$ time with high probability.

Because...

Theorem

Let X_n be the number heads when flipping a coin independently n times. Then

$$\Pr\left[X_n \leq \frac{n}{4}\right] \leq 2 \cdot 0.68^{n/4} \text{ and } \Pr\left[X_n \geq \frac{3n}{4}\right] \leq 2 \cdot 0.68^{n/4}$$

Randomized Quick Selection

Input Unsorted array **A** of **n** integers

Goal Find the **j**th smallest number in **A** (*rank j* number)

Randomized Quick Selection

- 1 Pick a pivot element *uniformly at random* from the array
- 2 Split array into 3 subarrays: those smaller than pivot, those larger than pivot, and the pivot itself.
- 3 Return pivot if rank of pivot is **j**.
- 4 Otherwise recurse on one of the arrays depending on **j** and their sizes.

Algorithm for Randomized Selection

Assume for simplicity that **A** has distinct elements.

QuickSelect(**A**, **j**):

Pick pivot **x** uniformly at random from **A**

Partition **A** into **A_{less}**, **x**, and **A_{greater}** using **x** as pivot

if (**|A_{less}|** = **j** - 1) **then**

return **x**

if (**|A_{less}|** ≥ **j**) **then**

return **QuickSelect**(**A_{less}**, **j**)

else

return **QuickSelect**(**A_{greater}**, **j** - **|A_{less}|** - 1)

QuickSelect analysis

- ① S_1, S_2, \dots, S_k be the subproblems considered by the algorithm. Here $|S_1| = n$.
- ② S_i would be **successful** if $|S_i| \leq (3/4) |S_{i-1}|$
- ③ Y_1 = number of recursive calls till first successful iteration. Clearly, total work till this happens is $O(Y_1 n)$.
- ④ n_i = size of the subproblem immediately after the $(i - 1)$ th successful iteration.
- ⑤ Y_i = number of recursive calls after the $(i - 1)$ th successful call, till the i th successful iteration.
- ⑥ Running time is $O(\sum_i n_i Y_i)$.

QuickSelect analysis

Example

S_i = subarray used in i th recursive call

$|S_i|$ = size of this subarray

Red indicates successful iteration.

Inst'	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9
$ S_i $	100	70	60	50	40	30	25	5	2
Succ'	$Y_1 = 2$		$Y_2 = 4$				$Y_3 = 2$		$Y_4 = 1$
$n_i =$	$n_1 = 100$		$n_2 = 60$				$n_3 = 25$		$n_4 = 2$

- 1 All the subproblems after $(i - 1)$ th successful iteration till i th successful iteration have size $\leq n_i$.
- 2 Total work: $O(\sum_i n_i Y_i)$.

QuickSelect analysis

Total work: $O(\sum_i n_i Y_i)$.

We have:

- ① $n_i \leq (3/4)n_{i-1} \leq (3/4)^{i-1}n$.
- ② Y_i is a random variable with geometric distribution
Probability of $Y_i = k$ is $1/2^i$.
- ③ $E[Y_i] = 2$.

As such, expected work is proportional to

$$\begin{aligned} E\left[\sum_i n_i Y_i\right] &= \sum_i E[n_i Y_i] \leq \sum_i E[(3/4)^{i-1} n Y_i] \\ &= n \sum_i (3/4)^{i-1} E[Y_i] = n \sum_{i=1} (3/4)^{i-1} 2 \leq 8n. \end{aligned}$$

QuickSelect analysis

Theorem

The expected running time of QuickSelect is $O(n)$.

QuickSelect analysis

Analysis via Recurrence

- 1 Given array \mathbf{A} of size n let $Q(\mathbf{A})$ be number of comparisons of randomized selection on \mathbf{A} for selecting rank j element.
- 2 Note that $Q(\mathbf{A})$ is a random variable
- 3 Let $\mathbf{A}_{\text{less}}^i$ and $\mathbf{A}_{\text{greater}}^i$ be the left and right arrays obtained if pivot is rank i element of \mathbf{A} .
- 4 Algorithm recurses on $\mathbf{A}_{\text{less}}^i$ if $j < i$ and recurses on $\mathbf{A}_{\text{greater}}^i$ if $j > i$ and terminates if $j = i$.

$$Q(\mathbf{A}) = n + \sum_{i=1}^{j-1} \Pr[\text{pivot has rank } i] Q(\mathbf{A}_{\text{greater}}^i) + \sum_{i=j+1}^n \Pr[\text{pivot has rank } i] Q(\mathbf{A}_{\text{less}}^i)$$

QuickSelect analysis

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Analyzing the Recurrence

As in **QuickSort** we obtain the following recurrence where $T(n)$ is the worst-case expected time.

$$T(n) \leq n + \frac{1}{n} \left(\sum_{i=1}^{j-1} T(n-i) + \sum_{i=j}^n T(i-1) \right).$$

Theorem

$$T(n) = O(n).$$

Proof.

(Guess and) Verify by induction (see next slide). □

Analyzing the recurrence

Theorem

$$T(n) = O(n).$$

Prove by induction that $T(n) \leq \alpha n$ for some constant $\alpha \geq 1$ to be fixed later.

Base case: $n = 1$, we have $T(1) = 0$ since no comparisons needed and hence $T(1) \leq \alpha$.

Induction step: Assume $T(k) \leq \alpha k$ for $1 \leq k < n$ and prove it for $T(n)$. We have by the recurrence:

$$\begin{aligned} T(n) &\leq n + \frac{1}{n} \left(\sum_{i=1}^{j-1} T(n-i) + \sum_{i=j^n} T(i-1) \right) \\ &\leq n + \frac{\alpha}{n} \left(\sum_{i=1}^{j-1} (n-i) + \sum_{i=j}^n (i-1) \right) \quad \text{by applying induction} \end{aligned}$$

Analyzing the recurrence

$$\begin{aligned}T(n) &\leq n + \frac{\alpha}{n} \left(\sum_{i=1}^{j-1} (n-i) + \sum_{i=j}^n (i-1) \right) \\&\leq n + \frac{\alpha}{n} ((j-1)(2n-j)/2 + (n-j+1)(n+j-2)/2) \\&\leq n + \frac{\alpha}{2n} (n^2 + 2nj - 2j^2 - 3n + 4j - 2) \\&\quad \text{above expression maximized when } j = (n+1)/2: \text{ calculus} \\&\leq n + \frac{\alpha}{2n} (3n^2/2 - n) \quad \text{substituting } (n+1)/2 \text{ for } j \\&\leq n + 3\alpha n/4 \\&\leq \alpha n \quad \text{for any constant } \alpha \geq 4\end{aligned}$$

Comments on analyzing the recurrence

- 1 Algebra looks messy but intuition suggest that the median is the hardest case and hence can plug $j = n/2$ to simplify without calculus
- 2 Analyzing recurrences comes with practice and after a while one can see things more intuitively

John Von Neumann:

Young man, in mathematics you don't understand things. You just get used to them.

Notes

Notes

