Shortest Path Algorithms

Lecture 4

January 26, 2013

Part I

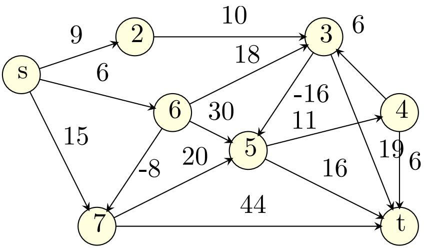
Shortest Paths with Negative Length Edges

Single-Source Shortest Paths with Negative Edge Lengths

Single-Source Shortest Path Problems

Input: A *directed* graph $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ with arbitrary (including negative) edge lengths. For edge $\mathbf{e} = (\mathbf{u}, \mathbf{v})$, $\ell(\mathbf{e}) = \ell(\mathbf{u}, \mathbf{v})$ is its length.

1. Given nodes $\mathbf{s}, \mathbf{t}$ find shortest path from $\mathbf{s}$ to $\mathbf{t}$.
2. Given node $\mathbf{s}$ find shortest path from $\mathbf{s}$ to all other nodes.



Single-Source Shortest Paths with Negative Edge Lengths

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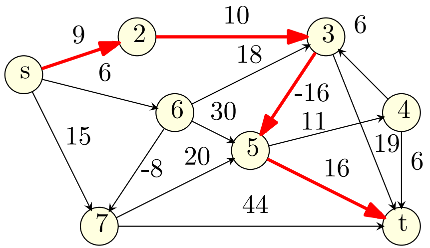
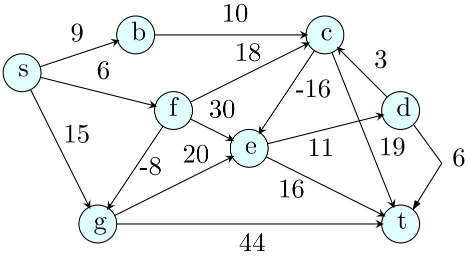


Negative Length Cycles

Definition

A cycle **C** is a negative length cycle if the sum of the edge lengths of **C** is negative.



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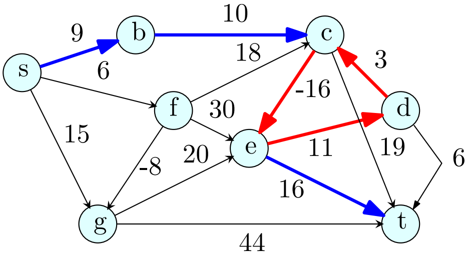

Shortest Paths and Negative Cycles

Given $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ with edge lengths and $\mathbf{s}, \mathbf{t}$. Suppose

1. $\mathbf{G}$ has a negative length cycle $\mathbf{C}$, and
2. $\mathbf{s}$ can reach $\mathbf{C}$ and $\mathbf{C}$ can reach $\mathbf{t}$.

Question: What is the shortest **distance** from $\mathbf{s}$ to $\mathbf{t}$?
Possible answers: Define shortest distance to be:

1. undefined, that is $-\infty$, OR
2. the length of a shortest **simple** path from $\mathbf{s}$ to $\mathbf{t}$.

Lemma

If there is an efficient algorithm to find a shortest simple $\mathbf{s} \rightarrow \mathbf{t}$ path in a graph with negative edge lengths, then there is an efficient algorithm to find the longest *simple $\mathbf{s} \rightarrow \mathbf{t}$ path in a graph with positive edge lengths.*

Finding the $\mathbf{s} \rightarrow \mathbf{t}$ longest path is difficult. **NP-Hard**!

Algorithmic Problems

Input: A directed graph $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ with arbitrary (including negative) edge lengths. For edge $\mathbf{e} = (\mathbf{u}, \mathbf{v})$, $\ell(\mathbf{e}) = \ell(\mathbf{u}, \mathbf{v})$ is its length.

Questions:

1. Given nodes $\mathbf{s}, \mathbf{t}$, either find a negative length cycle $\mathbf{C}$ that $\mathbf{s}$ can reach or find a shortest path from $\mathbf{s}$ to $\mathbf{t}$.
2. Given node $\mathbf{s}$, either find a negative length cycle $\mathbf{C}$ that $\mathbf{s}$ can reach or find shortest path distances from $\mathbf{s}$ to all reachable nodes.
3. Check if $\mathbf{G}$ has a negative length cycle or not.

Shortest Paths with Negative Edge Lengths
In Undirected Graphs

Note: With negative lengths, shortest path problems and negative cycle detection in undirected graphs cannot be reduced to directed graphs by bi-directing each undirected edge. Why?

Problem can be solved efficiently in undirected graphs but algorithms are different and more involved than those for directed graphs. Beyond the scope of this class. If interested, ask instructor for references.

Why Negative Lengths?

Several Applications

1. Shortest path problems useful in modeling many situations — in some negative lenths are natural
2. Negative length cycle can be used to find arbitrage opportunities in currency trading
3. Important sub-routine in algorithms for more general problem: minimum-cost flow

Currency Trading

Input: $\mathbf{n}$ currencies and for each ordered pair $\mathbf{(a, b)}$ the *exchange rate* for converting one unit of $\mathbf{a}$ into one unit of $\mathbf{b}$.

Questions:

1. Is there an arbitrage opportunity?
2. Given currencies $\mathbf{s}, \mathbf{t}$ what is the best way to convert $\mathbf{s}$ to $\mathbf{t}$ (perhaps via other intermediate currencies)?

Concrete example:

1. $\mathbf{1}$ Chinese Yuan $=$ $\mathbf{0.1116}$ Euro
2. $\mathbf{1}$ Euro $=$ $\mathbf{1.3617}$ US dollar
3. $\mathbf{1}$ US Dollar $=$ $\mathbf{7.1}$ Chinese Yuan.

Thus, if exchanging $\mathbf{1\ \$} \rightarrow$ Yuan $\rightarrow$ Euro $\rightarrow$ $\mathbf{\$}$, we get: $\mathbf{0.1116 * 1.3617 * 7.1 = 1.07896\$}$.

Reducing Currency Trading to Shortest Paths

Observation: If we convert currency $\mathbf{i}$ to $\mathbf{j}$ via intermediate currencies $\mathbf{k_1, k_2, \ldots, k_h}$ then one unit of $\mathbf{i}$ yields $\mathbf{exch(i, k_1) \times exch(k_1, k_2) \ldots \times exch(k_h, j)}$ units of $\mathbf{j}$.

Create currency trading *directed* graph $\mathbf{G = (V, E)}$:

1. For each currency $\mathbf{i}$ there is a node $\mathbf{v_i} \in \mathbf{V}$
2. $\mathbf{E = V \times V}$: an edge for each pair of currencies
3. edge length $\ell(\mathbf{v_i, v_j}) = -\mathbf{\log(exch(i, j))}$ can be negative

Exercise: Verify that

1. There is an arbitrage opportunity if and only if $\mathbf{G}$ has a negative length cycle.
2. The best way to convert currency $\mathbf{i}$ to currency $\mathbf{j}$ is via a shortest path in $\mathbf{G}$ from $\mathbf{i}$ to $\mathbf{j}$. If $\mathbf{d}$ is the distance from $\mathbf{i}$ to $\mathbf{j}$ then one unit of $\mathbf{i}$ can be converted into $\mathbf{2^d}$ units of $\mathbf{j}$.

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Reducing Currency Trading to Shortest Paths

Math recall - relevant information

1. $\log(\alpha_1 * \alpha_2 * \cdots * \alpha_k) = \log \alpha_1 + \log \alpha_2 + \cdots + \log \alpha_k$.
2. $\log x > 0$ if and only if $x > 1$.

Algorithmic Problems

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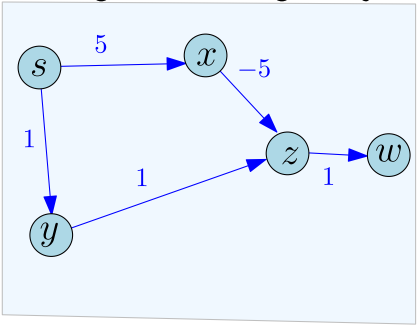
Dijkstra's Algorithm and Negative Lengths

With negative cost edges, Dijkstra's algorithm fails



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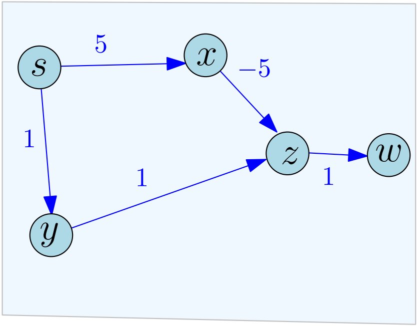

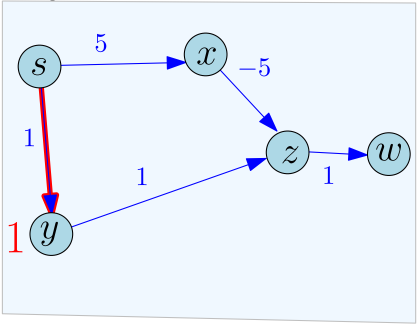

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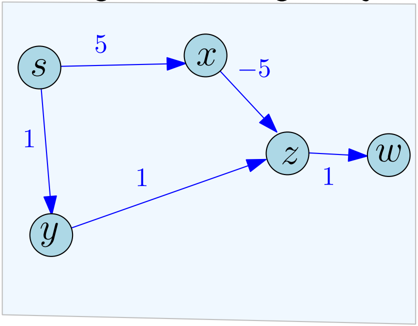

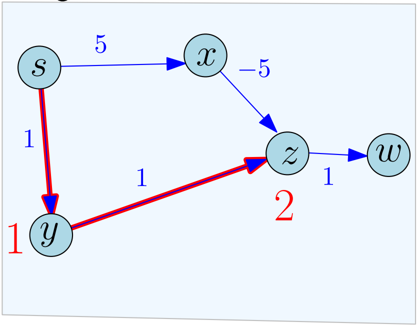

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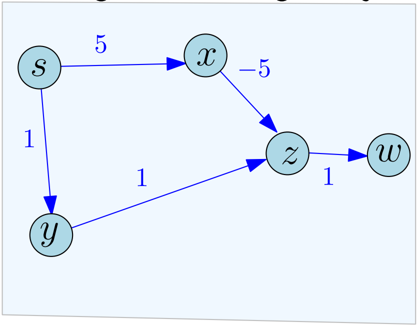

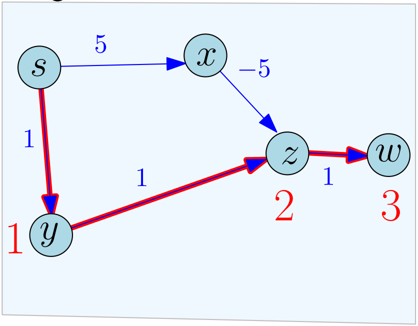

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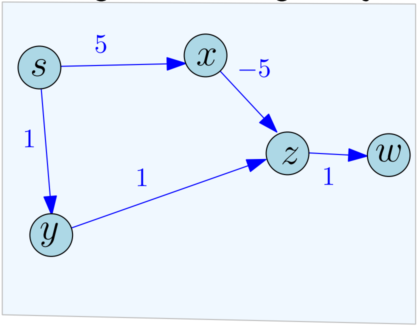

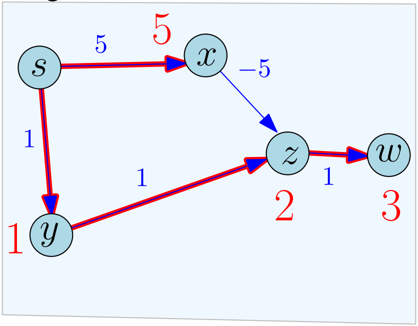

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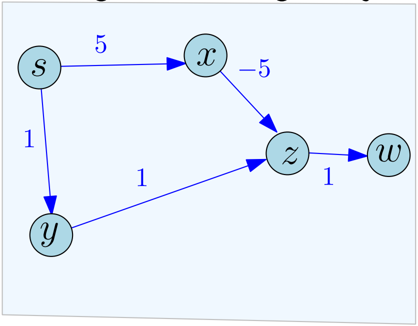

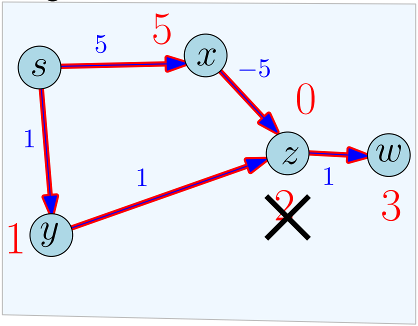

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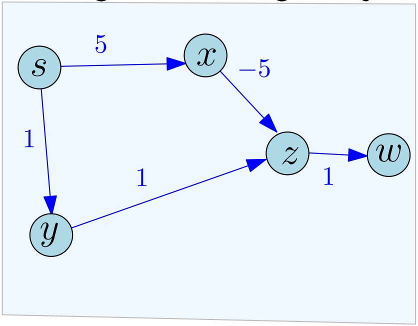

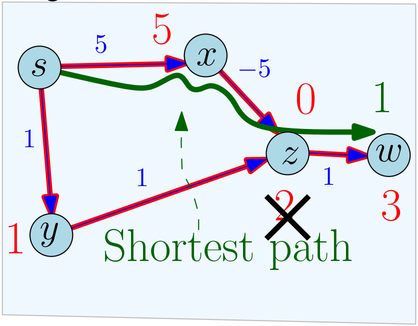

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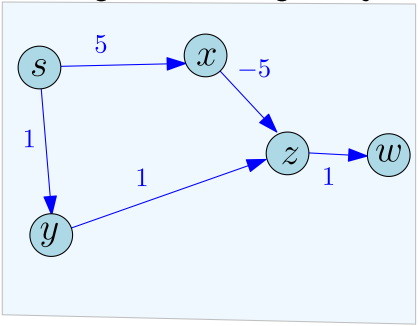

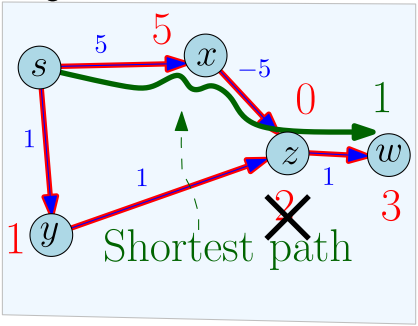

False assumption: Dijkstra's algorithm is based on the assumption that if $\mathbf{s} = \mathbf{v_0} \rightarrow \mathbf{v_1} \rightarrow \mathbf{v_2} \ldots \rightarrow \mathbf{v_k}$ is a shortest path from $\mathbf{s}$ to $\mathbf{v_k}$ then $\mathbf{dist}(\mathbf{s}, \mathbf{v_i}) \leq \mathbf{dist}(\mathbf{s}, \mathbf{v_{i+1}})$ for $\mathbf{0} \leq \mathbf{i} < \mathbf{k}$. Holds true only for non-negative edge lengths.

Shortest Paths with Negative Lengths

Lemma

Let $\mathbf{G}$ *be a directed graph with arbitrary edge lengths. If* $\mathbf{s} = \mathbf{v_0} \rightarrow \mathbf{v_1} \rightarrow \mathbf{v_2} \rightarrow \ldots \rightarrow \mathbf{v_k}$ *is a shortest path from* $\mathbf{s}$ *to* $\mathbf{v_k}$ *then for* $\mathbf{1} \leq \mathbf{i} < \mathbf{k}$*:*

1. $\mathbf{s} = \mathbf{v_0} \rightarrow \mathbf{v_1} \rightarrow \mathbf{v_2} \rightarrow \ldots \rightarrow \mathbf{v_i}$ *is a shortest path from* $\mathbf{s}$ *to* $\mathbf{v_i}$
2. *False:* $\mathbf{dist}(\mathbf{s}, \mathbf{v_i}) \leq \mathbf{dist}(\mathbf{s}, \mathbf{v_k})$ *for* $\mathbf{1} \leq \mathbf{i} < \mathbf{k}$*. Holds true only for non-negative edge lengths.*

Cannot explore nodes in increasing order of distance! We need a more basic strategy.

Shortest Paths with Negative Lengths

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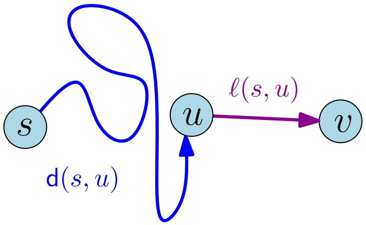
A Generic Shortest Path Algorithm

1. Start with distance estimate for each node $\mathbf{d(s, u)}$ set to ∞
2. Maintain the invariant that there is an $\mathbf{s} \rightarrow \mathbf{u}$ path of length $\mathbf{d(s, u)}$. Hence $\mathbf{d(s, u)} \geq \mathbf{dist(s, u)}$.
3. Iteratively refine $\mathbf{d(s, \cdot)}$ values until they reach the correct value $\mathbf{dist(s, \cdot)}$ values at termination

Must hold that...

$\mathbf{d(s, v)} \leq \mathbf{d(s, u)} + \ell(\mathbf{u}, \mathbf{v})$



A Generic Shortest Path Algorithm

Question: How do we make progress?

Definition

Given distance estimates $\mathbf{d(s,u)}$ for each $\mathbf{u} \in \mathbf{V}$, an edge $\mathbf{e} = \mathbf{(u,v)}$ is **tense** if $\mathbf{d(s,v)} > \mathbf{d(s,u)} + \ell(\mathbf{u},\mathbf{v})$.

Relax$(\mathbf{e} = (\mathbf{u},\mathbf{v}))$
    if $(\mathbf{d(s,v)} > \mathbf{d(s,u)} + \ell(\mathbf{u},\mathbf{v}))$ **then**
        $\mathbf{d(s,v)} = \mathbf{d(s,u)} + \ell(\mathbf{u},\mathbf{v})$

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A Generic Shortest Path Algorithm

Invariant

If a vertex $\mathbf{u}$ has value $\mathbf{d(s,u)}$ associated with it, then there is a $\mathbf{s} \rightsquigarrow \mathbf{u}$ walk of length $\mathbf{d(s,u)}$.

Proposition

Relax *maintains the invariant on* $\mathbf{d(s,u)}$ *values.*

Proof.

Indeed, if **Relax**$((\mathbf{u},\mathbf{v}))$ changed the value of $\mathbf{d(s,v)}$, then there is a walk to $\mathbf{u}$ of length $\mathbf{d(s,u)}$ (by invariant), and there is a walk of length $\mathbf{d(s,u)} + \ell(\mathbf{u},\mathbf{v})$ to $\mathbf{v}$ through $\mathbf{u}$, which is the new value of $\mathbf{d(s,v)}$. $\square$

A Generic Shortest Path Algorithm

$\mathbf{d(s, s) = 0}$
for each node $\mathbf{u \neq s}$ **do**
    $\mathbf{d(s, u) = \infty}$

while there is a tense edge **do**
    Pick a tense edge $\mathbf{e}$
    Relax($\mathbf{e}$)

Output $\mathbf{d(s, u)}$ values

Technical assumption: If $\mathbf{e = (u, v)}$ is an edge and $\mathbf{d(s, u) = d(s, v) = \infty}$ then edge is not tense.

Properties of the generic algorithm

Proposition

If $\mathbf{u}$ *is not reachable from* $\mathbf{s}$ *then* $\mathbf{d(s,u)}$ *remains at* ∞ *throughout the algorithm.*

Properties of the generic algorithm

Proposition

If a negative length cycle $\mathbf{C}$ *is reachable by* $\mathbf{s}$ *then there is always a tense edge and hence the algorithm never terminates.*

Proof Sketch.

Let $\mathbf{C} = \mathbf{v_0}, \mathbf{v_1}, \ldots, \mathbf{v_k}$ be a negative length cycle. Suppose algorithm terminates. Since no edge of $\mathbf{C}$ was tense, for $\mathbf{i} = 1, 2, \ldots, \mathbf{k}$ we have $\mathbf{d}(\mathbf{s}, \mathbf{v_i}) \leq \mathbf{d}(\mathbf{s}, \mathbf{v_{i-1}}) + \ell(\mathbf{v_{i-1}}, \mathbf{v_i})$ and $\mathbf{d}(\mathbf{s}, \mathbf{v_0}) \leq \mathbf{d}(\mathbf{s}, \mathbf{v_k}) + \ell(\mathbf{v_k}, \mathbf{v_0})$. Adding up all the inequalities we obtain that length of $\mathbf{C}$ is non-negative! $\square$

Corollary

If the algorithm terminates then there is no negative length cycle $\mathbf{C}$ *that is reachable from* $\mathbf{s}$.

Properties of the generic algorithm

Lemma

If the algorithm terminates then $\mathbf{d(s, u) = dist(s, u)}$ *for each node* $\mathbf{u}$ *(and* $\mathbf{s}$ *cannot reach a negative cycle).*

Proof of lemma; see future slides.

Properties of the generic algorithm

If estimate distance from source too large, then ∃ tense edge...

Lemma

If ∃ *walk* $\pi \equiv s = v_1 \to v_2 \to \cdots \to v_k = u$ *such that*

$$\ell(\pi) = \sum_{i=1}^{k-1} \ell(v_i, v_j) < d(s, u)$$

Then, there exists a tense edge in G.

Proof.

Assume π: shortest in number of edges (with property).

$\implies \ell(v_1 \to \cdots v_{k-1}) \geq d(s, v_{k-1})$.

$\implies d(s, v_{k-1}) + \ell(v_{k-1}, v_k)$

$\leq \ell(v_1 \to \cdots v_{k-1}) + \ell(v_{k-1}, v_k)$

$= \ell(\pi) < d(s, v_k)$.

$\implies d(s, v_{k-1}) + \ell(v_{k-1}, v_k) < d(s, v_k)$

$\implies$ edge (v_{k-1}, v_k) is tense. □

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$\implies \ell(v_1 \to \cdots v_{k-1}) \geq d(s, v_{k-1})$.

$\implies d(s, v_{k-1}) + \ell(v_{k-1}, v_k)$

$\leq \ell(v_1 \to \cdots v_{k-1}) + \ell(v_{k-1}, v_k)$

$= \ell(\pi) < d(s, v_k)$.

$\implies d(s, v_{k-1}) + \ell(v_{k-1}, v_k) < d(s, v_k)$

$\implies$ edge (v_{k-1}, v_k) is tense. □

Properties of the generic algorithm

If estimate distance from source too large, then ∃ tense edge...

Lemma

If ∃ *walk* $\pi \equiv s = v_1 \to v_2 \to \cdots \to v_k = u$ *such that*

$$\ell(\pi) = \sum_{i=1}^{k-1} \ell(v_i, v_j) < d(s, u)$$

Then, there exists a tense edge in G.

Proof.

Assume π: shortest in number of edges (with property).

$\implies \ell(v_1 \to \cdots v_{k-1}) \geq d(s, v_{k-1})$.

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$$\implies d(s, v_{k-1}) + \ell(v_{k-1}, v_k) \leq \ell(v_1 \rightarrow \cdots v_{k-1}) + \ell(v_{k-1}, v_k) = \ell(\pi) < d(s, v_k).$$

$\implies d(s, v_{k-1}) + \ell(v_{k-1}, v_k) < d(s, v_k)$

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Properties of the generic algorithm

If estimate distance from source too large, then $\exists$ tense edge...

Lemma

If $\exists$ *walk* $\pi \equiv \mathbf{s} = \mathbf{v_1} \to \mathbf{v_2} \to \cdots \to \mathbf{v_k} = \mathbf{u}$ *such that*

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Then, there exists a tense edge in $\mathbf{G}$.

Proof.

Assume π: shortest in number of edges (with property).

$\implies \ell(\mathbf{v_1} \to \cdots \mathbf{v_{k-1}}) \geq \mathbf{d}(\mathbf{s}, \mathbf{v_{k-1}})$.

$\implies \mathbf{d}(\mathbf{s}, \mathbf{v_{k-1}}) + \ell(\mathbf{v_{k-1}}, \mathbf{v_k})$
$\quad \leq \ell(\mathbf{v_1} \to \cdots \mathbf{v_{k-1}}) + \ell(\mathbf{v_{k-1}}, \mathbf{v_k})$
$\quad = \ell(\pi) < \mathbf{d}(\mathbf{s}, \mathbf{v_k})$.

$\implies \mathbf{d}(\mathbf{s}, \mathbf{v_{k-1}}) + \ell(\mathbf{v_{k-1}}, \mathbf{v_k}) < \mathbf{d}(\mathbf{s}, \mathbf{v_k})$

$\implies$ edge $(\mathbf{v_{k-1}}, \mathbf{v_k})$ is tense. $\square$

Properties of the generic algorithm

If estimate distance from source too large, then ∃ tense edge...

Lemma

If $\exists$ *walk* $\pi \equiv s = v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_k = u$ *such that*

$$\ell(\pi) = \sum_{i=1}^{k-1} \ell(v_i, v_j) < d(s, u)$$

Then, there exists a tense edge in G.

Proof.

Assume π: shortest in number of edges (with property).

$$\dots$$

$\implies$ edge (v_{k-1}, v_k) is tense. □

$\implies$ If for any vertex u: $d(s, u) > \mathrm{dist}(s, u)$ then the algorithm will continue working!

Generic Algorithm: Ordering Relax operations

```
d(s,s) = 0
for each node u ≠ s do
    d(s,u) = ∞

While there is a tense edge do
    Pick a tense edge e
    Relax(e)

Output d(s,u) values for u ∈ V(G)
```

Question: How do we pick edges to relax?

Observation: Suppose $s \to v_1 \to \ldots \to v_k$ is a shortest path.

If $\mathsf{Relax}(s, v_1)$, $\mathsf{Relax}(v_1, v_2)$, $\ldots$, $\mathsf{Relax}(v_{k-1}, v_k)$ are done in *order* then $d(s, v_k) = dist(s, v_k)$!

Generic Algorithm: Ordering Relax operations

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Ordering Relax operations

1. **Observation:** Suppose $\mathbf{s} \to \mathbf{v_1} \to \ldots \to \mathbf{v_k}$ is a shortest path.

   If **Relax**$(\mathbf{s}, \mathbf{v_1})$, **Relax**$(\mathbf{v_1}, \mathbf{v_2})$, $\ldots$, **Relax**$(\mathbf{v_{k-1}}, \mathbf{v_k})$ are done in *order* then $\mathbf{d(s, v_k) = dist(s, v_k)}$! (Why?)
2. We don't know the shortest paths so how do we know the order to do the Relax operations?

Ordering Relax operations

1. **Observation:** Suppose $\mathbf{s} \rightarrow \mathbf{v_1} \rightarrow \ldots \rightarrow \mathbf{v_k}$ is a shortest path.

   If $\mathbf{Relax}(\mathbf{s}, \mathbf{v_1})$, $\mathbf{Relax}(\mathbf{v_1}, \mathbf{v_2})$, $\ldots$, $\mathbf{Relax}(\mathbf{v_{k-1}}, \mathbf{v_k})$ are done in *order* then $\mathbf{d(s, v_k) = dist(s, v_k)}$! (Why?)
2. We don't know the shortest paths so how do we know the order to do the Relax operations?

Ordering Relax operations

1. We don't know the shortest paths so how do we know the order to do the Relax operations?
2. We don't!
   1. Relax *all* edges (even those not tense) in some arbitrary order
   2. Iterate $|V| - 1$ times
   3. First iteration will do **Relax**(s, v_1) (and other edges), second round **Relax**(v_1, v_2) and in iteration k we do **Relax**(v_{k-1}, v_k).

Ordering Relax operations

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Bellman-Ford Algorithm

for each $\mathbf{u} \in \mathbf{V}$ **do**
  $\mathbf{d(s,u)} \leftarrow \infty$
$\mathbf{d(s,s)} \leftarrow \mathbf{0}$

for $\mathbf{i} = \mathbf{1}$ to $|\mathbf{V}| - \mathbf{1}$ **do**
  for each edge $\mathbf{e} = (\mathbf{u}, \mathbf{v})$ **do**
    Relax($\mathbf{e}$)

for each $\mathbf{u} \in \mathbf{V}$ **do**
  $\mathrm{dist}(\mathbf{s}, \mathbf{u}) \leftarrow \mathbf{d(s,u)}$

Bellman-Ford Algorithm: Scanning Edges

One possible way to scan edges in each iteration.

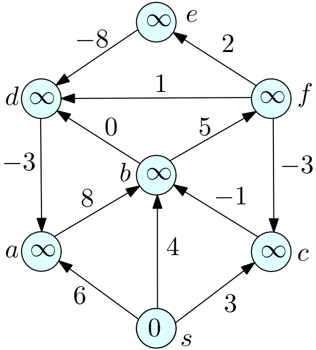
```
Q is an empty queue
for each u ∈ V do
        d(s, u) = ∞
        enq(Q, u)
d(s, s) = 0

for i = 1 to |V| − 1 do
        for j = 1 to |V| do
            u = deq(Q)
            for each edge e in Adj(u) do
                Relax(e)
            enq(Q, u)

for each u ∈ V do
        dist(s, u) = d(s, u)
```

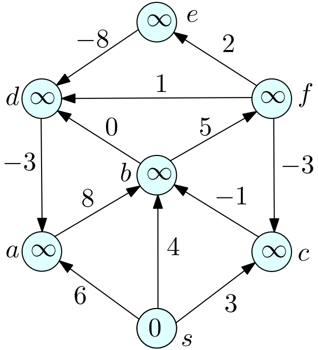
Example

Step 0



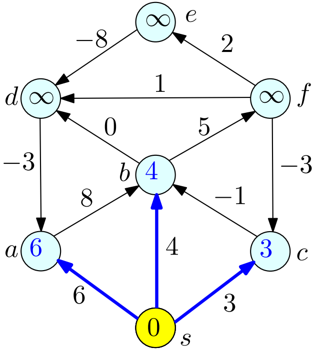
Example

Step 0

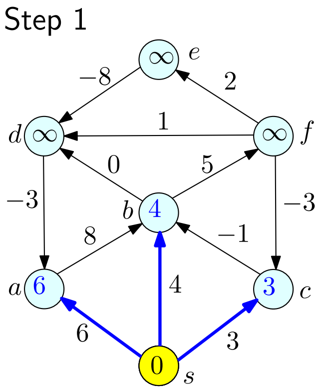


$\Longrightarrow$

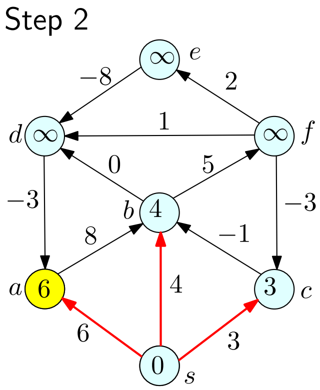
Step 1



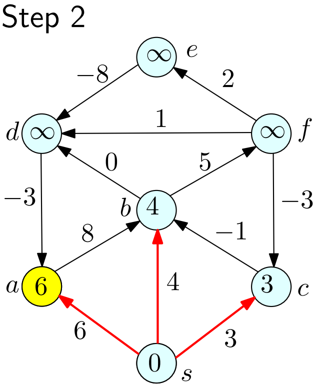
Example



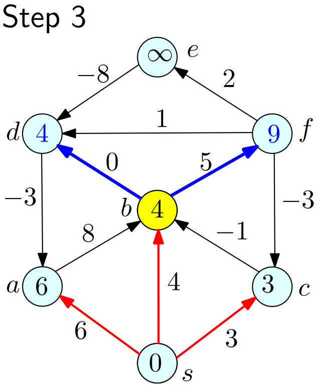
$\Longrightarrow$



Example

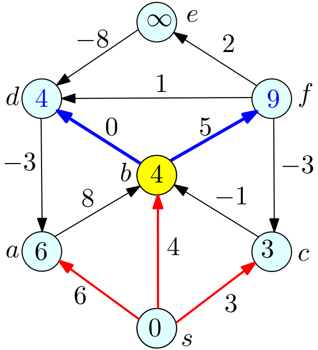


$\Longrightarrow$



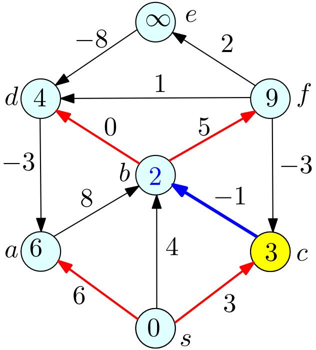
Example

Step 3

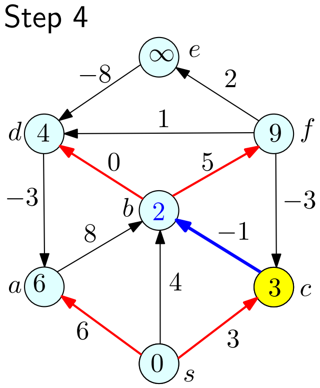


$\Longrightarrow$

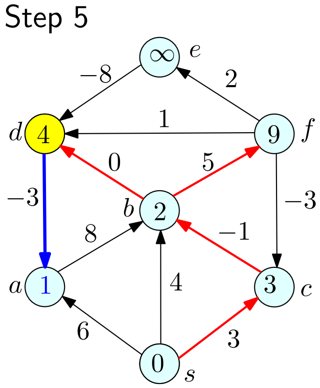
Step 4



Example



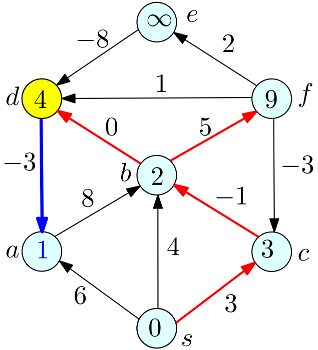
Step 4
∞
e
−8
2
d
4
1
9
f
0
5
−3
b
2
−3
8
−1
a
6
4
3
c
6
3
0
s



Step 5
∞
e
−8
2
d
4
1
9
f
0
5
−3
b
2
−3
8
−1
a
1
4
3
c
6
3
0
s

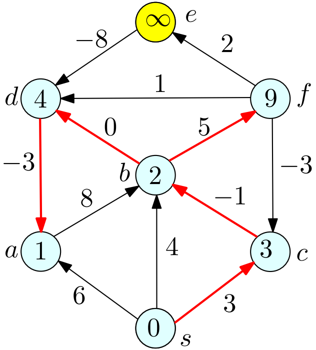
Example

Step 5

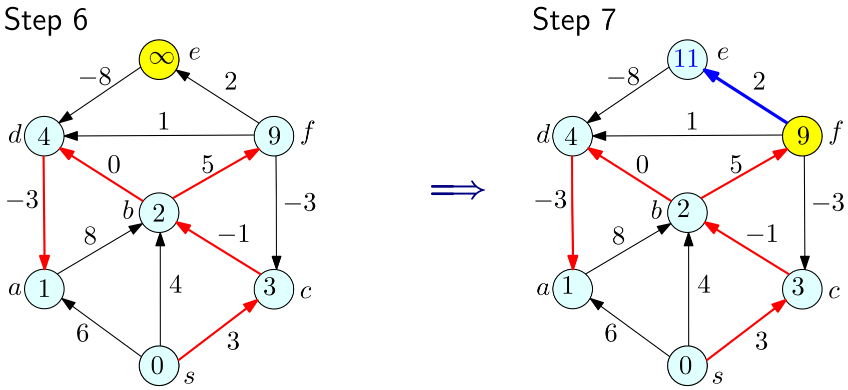


$\Longrightarrow$

Step 6



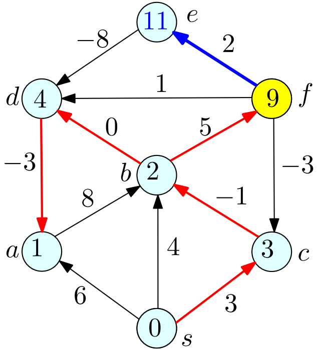
Example



Step 6
∞
e
−8
2
d
4
1
9
f
0
5
−3
b
2
−3
8
−1
a
1
4
3
c
6
3
0
s
Step 7
11
e
−8
2
d
4
1
9
f
0
5
−3
b
2
−3
8
−1
a
1
4
3
c
6
3
0
s

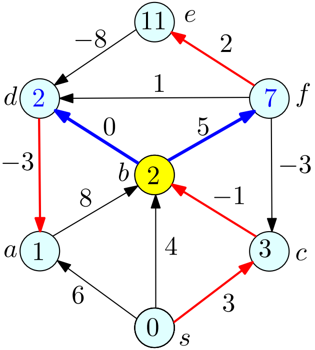
Example

Step 7



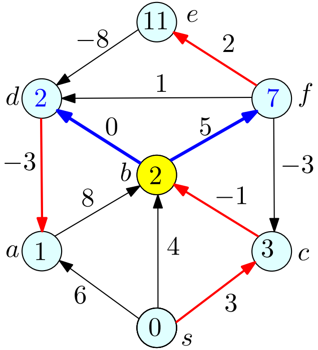
$\Longrightarrow$

Step 8



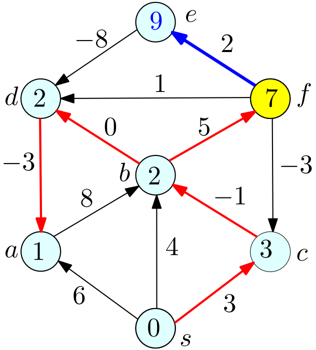
Example

Step 8



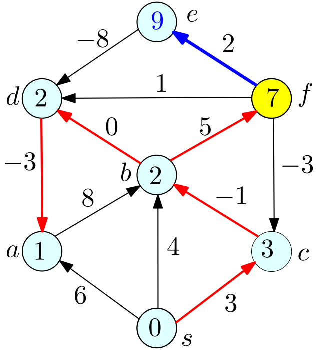
$\Longrightarrow$

Step 9

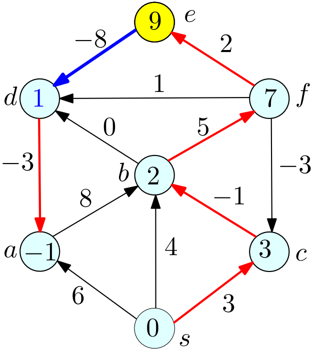


Example

Step 9

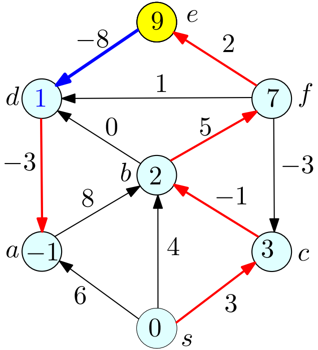


Step 10



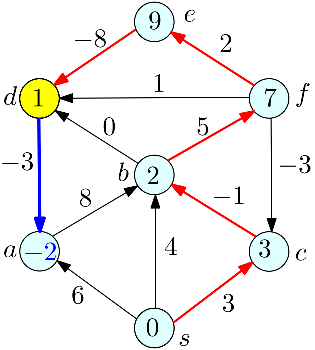
Example

Step 10



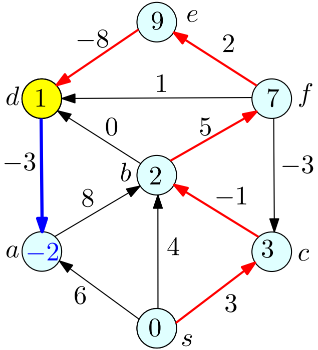
$\Longrightarrow$

Step 11



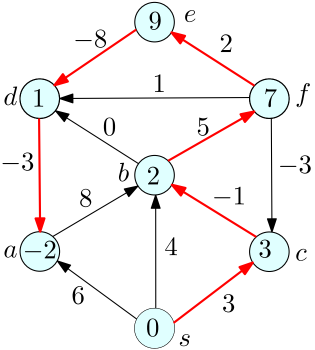
Example

Step 11



$\Longrightarrow$

Step 12



We are done! No edge is tense.

Example

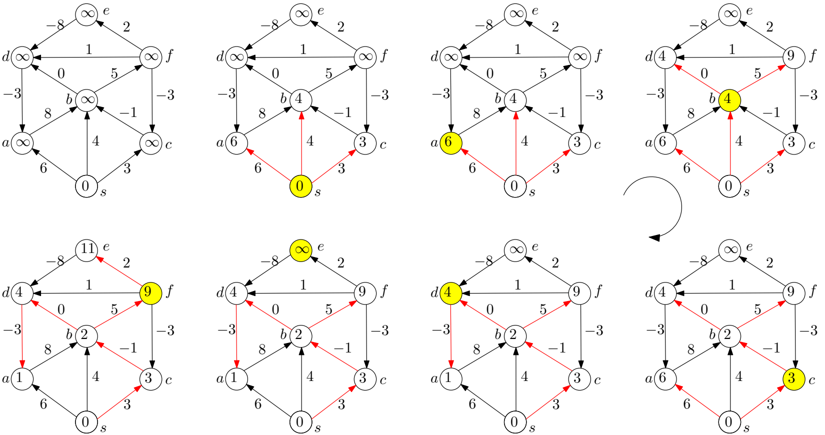


Figure: One iteration of Bellman-Ford that Relaxes all edges by processing nodes in the order **s, a, b, c, d, e, f**. Red edges indicate the prev pointers (in reverse)

Example

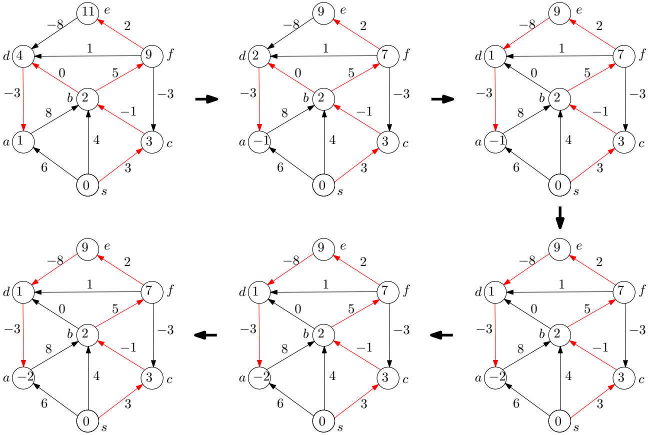


Figure: **6** iterations of Bellman-Ford starting with the first one from previous slide. No changes in **5**th iteration and **6**th iteration.

Correctness of the Bellman-Ford Algorithm

Lemma

G: *a directed graph with arbitrary edge lengths,* v: *a node in* V *s.t. there is a shortest path from* s *to* v *with* i *edges. Then, after* i *iterations of the loop in Bellman-Ford,* $d(s, v) = dist(s, v)$

Proof.

By induction on i.

1. Base case: $i = 0$. $d(s, s) = 0$ and $d(s, s) = dist(s, s)$.
2. Induction Step: Let $s \to v_1 \ldots \to v_{i-1} \to v$ be a shortest path from s to v of i hops.
   1. v_{i-1} has a shortest path from s of $i - 1$ hops or less. (Why?). By induction, $d(s, v_{i-1}) = dist(s, v_{i-1})$ after $i - 1$ iterations.
   2. In iteration i, **Relax**(v_{i-1}, v_i) sets $d(s, v_i) = dist(s, v_i)$.
   3. Note: Relax does not change $d(s, u)$ once $d(s, u) = dist(s, u)$.

□

Correctness of Bellman-Ford Algorithm

Corollary

After $|\mathbf{V}| - 1$ *iterations of Bellman-Ford,* $\mathbf{d(s, u) = dist(s, u)}$ *for any node* $\mathbf{u}$ *that has a shortest path from* $\mathbf{s}$.

Note: If there is a negative cycle $\mathbf{C}$ such that $\mathbf{s}$ can reach $\mathbf{C}$ then we do not know whether $\mathbf{d(s, u) = dist(s, u)}$ or not even if $\mathbf{dist(s, u)}$ is well-defined.

Question: How do we know whether there is a negative cycle $\mathbf{C}$ reachable from $\mathbf{s}$?

Correctness of Bellman-Ford Algorithm

Corollary

After $|\mathbf{V}| - 1$ *iterations of Bellman-Ford,* $\mathbf{d(s,u) = dist(s,u)}$ *for any node* $\mathbf{u}$ *that has a shortest path from* $\mathbf{s}$*.*

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After $|\mathbf{V}| - 1$ *iterations of Bellman-Ford,* $\mathbf{d(s, u) = dist(s, u)}$ *for any node* $\mathbf{u}$ *that has a shortest path from* $\mathbf{s}$*.*

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Question: How do we know whether there is a negative cycle $\mathbf{C}$ reachable from $\mathbf{s}$?

Bellman-Ford to detect Negative Cycles

```
for each u ∈ V do
        d(s, u) = ∞
d(s, s) = 0

for i = 1 to |V| − 1 do
        for each edge e = (u, v) do
            Relax(e)

for each edge e = (u, v) do
        if e = (u, v) is tense then
            Stop and output that s can reach
                    a negative length cycle

Output for each u ∈ V:  d(s, u)
```

Correctness

Lemma

$\mathbf{G}$ has a negative cycle reachable from $\mathbf{s}$ if and only if there is a tense edge $\mathbf{e}$ after $|\mathbf{V}| - \mathbf{1}$ iterations of Bellman-Ford.

Proof Sketch.

$\mathbf{G}$ has no negative length cycle reachable from $\mathbf{s}$ implies that all nodes $\mathbf{u}$ have a shortest path from $\mathbf{s}$. Therefore $\mathbf{d(s,u) = dist(s,u)}$ after the $|\mathbf{V}| - \mathbf{1}$ iterations. Therefore, there cannot be any tense edges left.

If there is a negative cycle $\mathbf{C}$ then there is a tense edge after $|\mathbf{V}| - \mathbf{1}$ (in fact any number of) iterations. See lemma about properties of the generic shortest path algorithm. □

Correctness

Lemma

$\mathbf{G}$ *has a negative cycle reachable from* $\mathbf{s}$ *if and only if there is a tense edge* $\mathbf{e}$ *after* $|\mathbf{V}| - \mathbf{1}$ *iterations of Bellman-Ford.*

Proof Sketch.

$\mathbf{G}$ has no negative length cycle reachable from $\mathbf{s}$ implies that all nodes $\mathbf{u}$ have a shortest path from $\mathbf{s}$. Therefore $\mathbf{d(s, u) = dist(s, u)}$ after the $|\mathbf{V}| - \mathbf{1}$ iterations. Therefore, there cannot be any tense edges left.

If there is a negative cycle $\mathbf{C}$ then there is a tense edge after $|\mathbf{V}| - \mathbf{1}$ (in fact any number of) iterations. See lemma about properties of the generic shortest path algorithm. □

Finding the Paths and a Shortest Path Tree

```
for each u ∈ V do
        d(s, u) = ∞
        prev(u) = null
d(s, s) = 0
for i = 1 to |V| − 1 do
        for each edge e = (u, v) do
            Relax(e)
if there is a tense edge e then
        Output that s can reach a negative cycle C
else
        for each u ∈ V do
            output d(s, u)

Relax(e = (u, v))
        if (d(s, v) > d(s, u) + ℓ(u, v)) then
            d(s, v) = d(s, u) + ℓ(u, v)
            prev(v) = u
```

Note: prev pointers induce a shortest path tree.

Negative Cycle Detection

Negative Cycle Detection

Given directed graph G with arbitrary edge lengths, does it have a negative length cycle?

1. Bellman-Ford checks whether there is a negative cycle C that is reachable from a specific vertex s. There may negative cycles not reachable from s.
2. Run Bellman-Ford $|V|$ times, once from each node u?

Negative Cycle Detection

Negative Cycle Detection

Given directed graph $\mathbf{G}$ with arbitrary edge lengths, does it have a negative length cycle?

1. Bellman-Ford checks whether there is a negative cycle $\mathbf{C}$ that is reachable from a specific vertex $\mathbf{s}$. There may negative cycles not reachable from $\mathbf{s}$.
2. Run Bellman-Ford $|\mathbf{V}|$ times, once from each node $\mathbf{u}$?

Negative Cycle Detection

1. Add a new node $\mathbf{s'}$ and connect it to all nodes of $\mathbf{G}$ with zero length edges. Bellman-Ford from $\mathbf{s'}$ will fill find a negative length cycle if there is one. Exercise: why does this work?
2. Negative cycle detection can be done with one Bellman-Ford invocation.

Running time for Bellman-Ford

1. Input graph $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ with $\mathbf{m} = |\mathbf{E}|$ and $\mathbf{n} = |\mathbf{V}|$.
2. $\mathbf{n}$ outer iterations and $\mathbf{m}$ **Relax**() operations in each iteration. Each **Relax**() operation is $\mathbf{O(1)}$ time.
3. Total running time: $\mathbf{O(mn)}$.

Dijkstra's Algorithm with **Relax**()

```
for each node u ≠ s do
    d(s, u) = ∞
d(s, s) = 0
S = ∅
while (S ≠ V) do
    Let v be node in V − S with min d value
    S = S ∪ {v}
    for each edge e in Adj(v) do
        Relax(e)
```

Part II

Shortest Paths in DAGs

Shortest Paths in a DAG

Single-Source Shortest Path Problems

Input A directed acyclic graph $G = (V, E)$ with arbitrary (including negative) edge lengths. For edge $e = (u, v)$, $\ell(e) = \ell(u, v)$ is its length.

1. Given nodes s, t find shortest path from s to t.
2. Given node s find shortest path from s to all other nodes.

Simplification of algorithms for DAGs

1. No cycles and hence no negative length cycles! Hence can find shortest paths even for negative length edges
2. Can order nodes using topological sort

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Algorithm for DAGs

1. Want to find shortest paths from s. Ignore nodes not reachable from s.
2. Let $s = v_1, v_2, v_{i+1}, \ldots, v_n$ be a topological sort of G

Observation:

1. shortest path from s to v_i cannot use any node from $v_{i+1}, \ldots, v_n$
2. can find shortest paths in topological sort order.

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Algorithm for DAGs

for $\mathbf{i = 1}$ `to` $\mathbf{n}$ **do**
    $\mathbf{d(s, v_i) = \infty}$
$\mathbf{d(s, s) = 0}$

for $\mathbf{i = 1}$ `to` $\mathbf{n - 1}$ **do**
    for `each edge` $\mathbf{e}$ `in` $\mathbf{Adj(v_i)}$ **do**
        Relax($\mathbf{e}$)

return $\mathbf{d(s, \cdot)}$ `values computed`

Correctness: induction on $\mathbf{i}$ and observation in previous slide.
Running time: $\mathbf{O(m + n)}$ time algorithm! Works for negative edge lengths and hence can find *longest* paths in a DAG.

Takeaway Points

1. Shortest paths with potentially negative length edges arise in a variety of applications. Longest simple path problem is difficult (no known efficient algorithm and **NP-Hard**). We restrict attention to shortest walks and they are well defined only if there are no negative length cycles reachable from the source.
2. A generic shortest path algorithm starts with distance estimates to the source and iteratively refines them by considering edges one at a time. The algorithm is guaranteed to terminate with correct distances if there are no negative length cycle. If a negative length cycle is reachable from the source it is guaranteed not to terminate.
3. Dijkstra's algorithm can also be thought of as an instantiation of the generic algorithm.

Points continued

1. Bellman-Ford algorithm is an instantiation of the generic algorithm that in each iteration relaxes all the edges. It recognizes negative length cycles if there is a tense edges in the nth iteration. For a vertex u with a shortest path to the source with i edges the algorithm has the correct distance after i iterations. Running time of Bellman-Ford algorithm is $O(nm)$.
2. Bellman-Ford can be adapted to find a negative length cycle in the graph by adding a new vertex.
3. If we have a DAG then it has no negative length cycle and hence shortest paths exists even with negative lengths. One can compute single-source shortest paths in a DAG in linear time. This implies that one can also compute longest paths in a DAG in linear time.

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