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CS473 Algorithms: lecture 12 (2024-02-27)

logistics - pset5 out F17

last lecture - flows
- reductions to max flow [forward; solve max flow backwards] [bipartite matchings]
- circulations w/ demands [reduce to max flow]
- ~~circulations~~ w/ lower-bounds [reduce to no lower bounds]
- bipartite matching w/ forced matches [lies in max flow]

today: randomized algorithms

reading: 13.0, 13.1, 13.12

Q - what is the "most realistic" model of computation? - runs on actual computers - captures abilities of actual computers

↳ randomized algo can use rand(k) - O(1) time - returns $0 \le i < k$, w/p $\frac{1}{k}$ [↳ rand()? [uniform dist]]

[Q. how does this help?]

Q - who wins the next election? [always same election] ["compressed" problem]

A - ask everyone [like, voting]
- ~328 million ppl
- ~235 million eligible voters
- ~158 [actual] voters [sad]

a lot [expensive] [do faster?]

fact [familiar] poll of 738 uniformly random voters will [good, often enough]
- estimate actual vote to error ≤ 5%
- with probability > 95%

[uniform sampling is hard] [response bias] [faster!] [is relaxed tradeoff] [will see proof]

Q - why rand algo?

A: can be simpler, [but] analysis can be more complicated

can be faster, [but] may fail with some probability [will see]

sometimes [only known] efficient algo is randomized

Q - how to model randomized algo?

models - deterministic - deterministic algo f

worst case input x

$x \mapsto f(x)$

[what is randomness?] [quick recap; please review]

complexity $T(n) = \max_{|x|=n} T(x)$ [worst case, function of input]

def - Ω finite / countably-infinite, $Pr: \Omega \to [0,1]$. Then (Ω, Pr) is a **discrete probability space** if $\sum_{\omega \in \Omega} Pr[\omega] = 1$

event $E \subseteq \Omega$, $Pr[E] = \sum_{\omega \in E} Pr[\omega]$

A random variable is a function $X: \Omega \to \mathbb{R}$, has expectation $\mathbb{E}[X] = \sum_{\omega \in \Omega} X(\omega) Pr[\omega]$

models - probabilistic - deterministic algo f

input $x \leftarrow \mathcal{D}_n$ ← distribution of size n inputs

$x \mapsto f(x)$

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complexity $T(n) = \mathbb{E}_{x \leftarrow D_n}[T(x)]$ [random var]

eg: machine learning [cats vs dogs]

rmk: theory is brittle

mfarbsign

randomized algo: $\boxed{\text{randomized}}$ algo f [$\boxed{\text{internal}}$ randomness]

worst case input x

$x \mapsto f(x)$ ← random variable

complexity: $T(n) = \max_{|x|=n} \mathbb{E}[T(x)]$ [worst case]

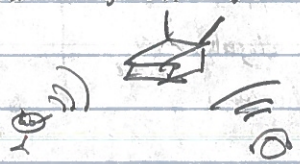
rmk: "right" worst case notion

-later

analysis: correctness: output correct $\boxed{\text{always}}$ [later: generalize]

complexity: $\boxed{\text{expected}}$ runtime [function of $|x|$] [later: generalize]

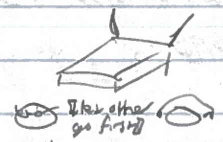
Q: how does wifi work?



→ collision! [no communication possible during collisions]

Q: how to avoid collisions?

idea: take turns



round 1	[silence]	[silence]	
round 2	)))	)))	→ collision?

idea: $\boxed{\text{assign}}$ turns [catch 22]

↳ requires $\boxed{\text{communication}}$

Q: how to assign turns $\boxed{\text{without}}$ centralized setup?

ie break symmetry

idea: use $\boxed{\text{names}}$ miforbes ← even length ⇒ talk in even rounds } no collisions!

alice ← odd length ⇒ talk in odd rounds

$\boxed{\text{but}}$ needs more work - bob ← ⇒ collisions

- names require $\boxed{\text{centralized}}$ setup

idea: use randomness!

def: the contention resolution process is, for n people $P_1, \ldots, P_n$

in round j: each person tries to communicate with probability p

if P_i is $\boxed{\text{only}}$ person communicating, they $\boxed{\text{succeed}}$

else, everyone fails [collision]

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Q: how many rounds until each person succeeds? [once everyone has communicated, we can assign turns] [start w/ $\boxed{\text{one}}$ person] [indicator random var]

def: $X_{ij} = \begin{cases} 1 & P_i \text{ attempts communication in round } j \\ 0 & \text{else} \end{cases}$

$\{X_{ij}\}_{ij}$ are $\boxed{\text{independent}}$ $\Pr[X_{ij} = \underset{\{0,1\}}{a} \wedge X_{i'j'} = \underset{\{0,1\}}{b}] = \Pr[X_{ij} = a] \cdot \Pr[X_{i'j'} = b]$ $X_{i'j'}$

lem: $\mathbb{E}[X_{ij}] = 1 \cdot \Pr[X_{ij} = 1] + 0 \cdot \Pr[X_{ij} = 0]$
$= \Pr[X_{ij} = 1]$ [true for $\boxed{\text{any}}$ indicator rand var]
$= p$

lem: $\mathbb{E}[\#\text{ people trying to communicate in round } j]$
$= \mathbb{E}[\sum_i X_{ij}]$
$= \sum_i \mathbb{E}[X_{ij}]$

lem [linearity of expectation] [!] $\boxed{\text{any}}$ rand var X, Y, $\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$

$= n \cdot p$

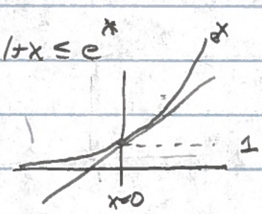
def: choose $p = 1/n$, so $= 1$ [want 1 person per round, so reasonable] [can also motivate formally]

lem: $\{P_i \text{ succeeds in round } j\} =$ [event] $\{X_{ij} = 1\}$ [P_i communicates] $\cap \{X_{i'j} = 0\}_{i' \neq i}$ [no one else tries]

cor: $\Pr[P_i \text{ succeeds round } j] = \Pr[X_{ij} = 1 \wedge \bigwedge_{i' \neq i} X_{i'j} = 0]$
$\overset{\text{indep}}{=} \underbrace{\Pr[X_{ij} = 1]}_{p} \cdot \prod_{i' \neq i} \underbrace{\Pr[X_{i'j} = 0]}_{1-p}$
$= 1/n (1 - 1/n)^{n-1}$ ☐

[how to understand]
idea: avoid $\boxed{\text{exact}}$ expressions
- exact expressions often difficult to obtain
- " to utilize, eg. [easy here] [hard in general] vs $1/100$

[inequalities]
lem [calculus]: $\forall x \in \mathbb{R}$, $1+x \leq e^x$
idea: - taylor expansion vs sketch



[not losing much in using this]

rmk: $1+x \approx e^x$ for $x \approx 0$

cor: $(1 - 1/n)^{n-1} \geq 1/e$, $n > 1$

pf: $\frac{1}{(1-1/n)^{n-1}} = \frac{1}{(\frac{n-1}{n})^{n-1}} = \left(\frac{n}{n-1}\right)^{n-1} = \left(1 + \frac{1}{n-1}\right)^{n-1} \leq (e^{1/n-1})^{n-1} = e$ [$x = \frac{1}{n-1} \geq 0$] ☐

cor: $\Pr[P_i \text{ succeeds in round } j] = \frac{1}{n} \underbrace{(1 - 1/n)^{n-1}}_{\geq 1/e} \geq 1/en$ [poor!] [$\Rightarrow$ do $\underline{\text{many}}$ rounds]

cor: $\Pr[P_i \text{ fails in rounds } 1, \ldots, t] \overset{\text{[indep]}}{=} \prod_{j=1}^{t} \Pr[P_i \text{ fails in round } j] \leq \left(1 - \frac{1}{en}\right)^t \leq e^{-t/en}$

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cor. $\Pr[P_i \text{ fails rounds } 1, \ldots, c \cdot en] \leq 1/e^c$

$\implies$ in $\Theta(n)$ rounds P_i succeeds w/p $1 - 1/e^{\Theta(1)} = 1 - \Theta(1) = \Theta(1)$

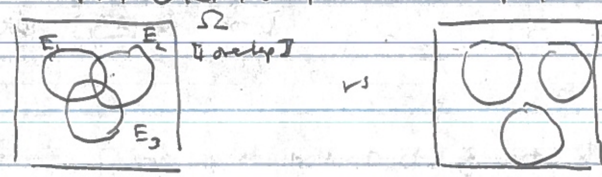
$\implies$ in $\Theta(n \ln n)$ rounds —"— $1 - 1/e^{\Theta(\ln n)} = 1 - 1/n^{\Theta(1)}$ [[high!]]

Q: $\boxed{\text{all}}$ players succeed?

lem [union bound] - events $E_1, \ldots, E_n$ [[possibly overlapping]]

$$\Pr[\cup_{i=1}^n E_i] \leq \sum_{i=1}^n \Pr[E_i]$$

sketch:



$\square$

cor. $\Pr[\boxed{\text{any}}\ P_i \text{ fails in rounds } 1, \ldots, c \cdot en]$

$= \Pr[\cup_i P_i \text{ fails in —"—}]$

$\overset{\text{union}}{\leq} \sum_i \Pr[P_i \text{ fails in —"—}]$

$\leq n \cdot 1/e^c$

cor: $\Pr[\text{any } P_i \text{ fails in rounds } 1, \ldots, (c+1) \cdot \ln n \cdot en] \leq 1/n^c$ [[high]]

$\implies \Pr[\text{all } P_i \text{ communicate in first } 2en \ln n \text{ rounds}] \geq 1 - 1/n$ [[many rounds unlikely]]

[[$n \ln n$ not much worse]]

rmk: - n rounds required [[only slightly more work]]

- can also analyze (expected) # rounds

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today - randomized algs
- motivation
- def
- contention resolution

reading - KT 13.0, 13.1, 13.12

next lecture - rand algs

logistics - pset5 due F17