ODE Stability

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- Solve the Ordinary Differential Equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \lambda\, y, \qquad \text{with} \qquad y\,(0) = y_0$$

- Exact Solution

$$y\,(t) = y_0\, e^{\lambda\, t}$$

- Solution is *asymptotically stable* or *decaying in magnitude* if

$$\mathrm{Re}(\lambda) < 0$$

- Solution is *stable* or *not growing in magnitude* if

$$\mathrm{Re}(\lambda) \leq 0$$

"Stable" values of λ in Complex Plane:

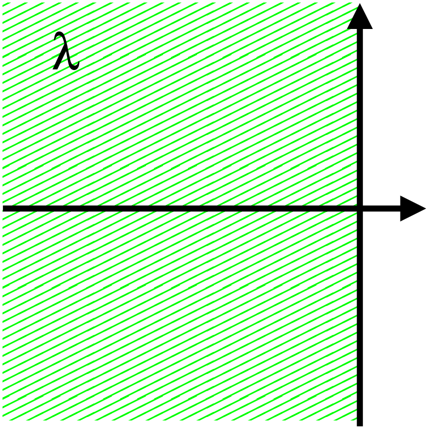
$$\lim_{t \to \infty} e^{\lambda t} = \quad ?$$

$\mathrm{Re}\,(\lambda) > 0 \quad$ (Unstable)

$\mathrm{Re}\,(\lambda) \leq 0 \quad$ (Stable)

Left Half-Plane = Stable

Right Half-Plane = Unstable



- Euler's Method

$$y_{k+1} = y_k + h\, f\,(t_k, y_k)$$

- Apply Euler's Method to

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \lambda\, y, \qquad \text{with} \qquad y\,(0) = y_0$$

we get

$$y_{k+1} = y_k + h\,\lambda y_k$$

- Regrouping terms

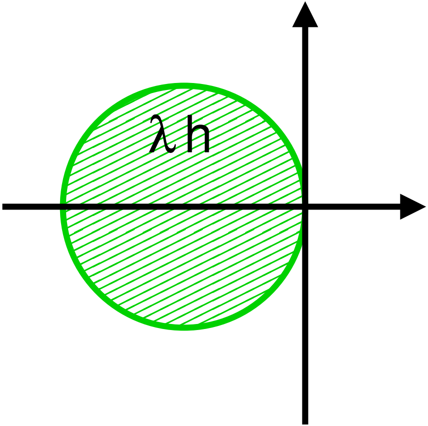
$$y_{k+1} = (1 + h\lambda)y_k = (1 + h\lambda)^{k+1} y_0$$

- Euler's Method is *stable* if

$$|1 + h\lambda| \leq 1$$

"Stable" values of $h\lambda$ in Complex Plane:

$$\lim_{k\to\infty} |1 + h\lambda|^k = \quad ?$$



$|1 + h\lambda| > 1$ (Unstable)

$|1 + h\lambda| \leq 1$ (Stable)

For $\mathrm{Re}(\lambda) < 0$, method is Conditionally Stable

For $\mathrm{Re}(\lambda) > 0$, method is Unconditionally Unstable

- Backward Euler's Method

$$y_{k+1} = y_k + h\,f\left(t_{k+1}, y_{k+1}\right)$$

- Apply Backward Euler's Method to

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \lambda\,y, \qquad \text{with} \qquad y\left(0\right) = y_0$$

we get

$$y_{k+1} = y_k + h\,\lambda y_{k+1}$$

- Regrouping terms

$$y_{k+1} = (1 - h\lambda)^{-1} y_k = \left(\frac{1}{1 - h\lambda}\right)^{k+1} y_0$$

- Backward Euler's Method is *stable* if

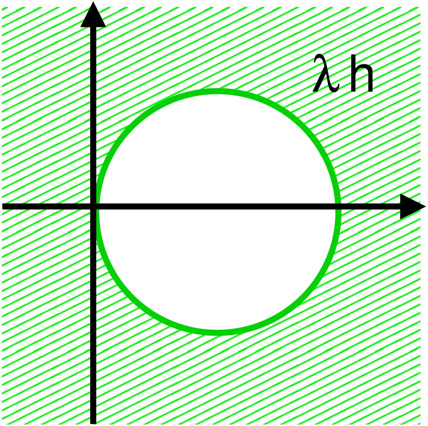
$$\frac{1}{|1 - h\lambda|} \leq 1$$

"Stable" values of $h\lambda$ in Complex Plane:

$$\lim_{k\to\infty} \frac{1}{|1 - h\lambda|^k} = \quad ?$$

$|1 - h\lambda| < 1$ (Unstable)

$|1 - h\lambda| \geq 1$ (Stable)



For $\mathrm{Re}(\lambda) < 0$, method is Unconditionally Stable

For $\mathrm{Re}(\lambda) > 0$, method is Conditionally Stable or Conditionally Unstable

- Trapezoid Method

$$y_{k+1} = y_k + \frac{h}{2} f(t_k, y_k) + \frac{h}{2} f(t_{k+1}, y_{k+1})$$

- Apply Trapezoid Method to

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \lambda y, \qquad \text{with} \qquad y(0) = y_0$$

we get

$$y_{k+1} = y_k + \frac{h}{2} \lambda y_k + \frac{h}{2} \lambda y_{k+1}$$

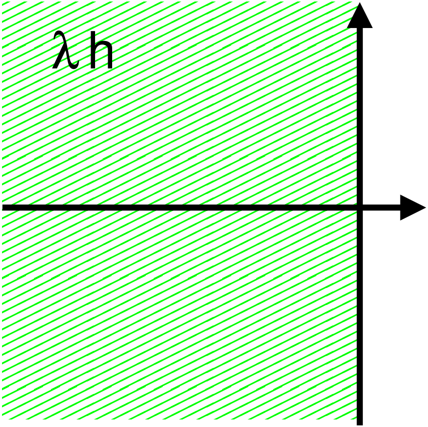
- Regrouping terms

$$y_{k+1} = \frac{1 + h\lambda/2}{1 - h\lambda/2} y_k = \left(\frac{1 + h\lambda/2}{1 - h\lambda/2}\right)^{k+1} y_0$$

- Trapezoid Method is *stable* if

$$\left|\frac{1 + h\lambda/2}{1 - h\lambda/2}\right| \leq 1$$

"Stable" values of $h\lambda$ in Complex Plane:



$$\lim_{k\to\infty}\left|\frac{1+h\lambda/2}{1-h\lambda/2}\right|^k = \quad ?$$

$|1+h\lambda/2| > |1-h\lambda/2|$ (Unstable)

$|1+h\lambda/2| \leq |1-h\lambda/2|$ (Stable)

For $\mathrm{Re}(\lambda) < 0$, method is Unconditionally Stable

For $\mathrm{Re}(\lambda) > 0$, method is Unconditionally Unstable

Linear Stability Summary

	$\mathrm{Re}\,(\lambda) <= 0$	$\mathrm{Re}\,(\lambda) > 0$
Exact Soln.	Stable	Unstable
Forward Euler	"Conditionally Stable" (stable for $\|1 + h\lambda\| \leq 1$)	"Unconditionally Unstable" (unstable for $h > 0$)
Backward Euler	"Unconditionally Stable" (stable for $h \geq 0$)	"Conditionally Stable" (stable for $\|1 - h\lambda\| \geq 1$)
Trapezoid	"Unconditionally Stable" (stable for $h \geq 0$)	"Unconditionally Unstable" (unstable for $h > 0$)

- Question: What about non-linear problems?

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t, y) \qquad \text{with} \qquad y(0) = y_0$$

- Answer: Use the derivative of f:

$$\text{Let} \quad \lambda = \frac{\partial f}{\partial y} \quad \text{and check stability}$$

for maximum and minimum values of λ.

- Question: What about linear systems of ODEs?

$$\frac{\mathrm{d}\vec{y}}{\mathrm{d}t} = A\,\vec{y} + g\,(t) \qquad \text{with} \qquad \vec{y}\,(0) = \vec{y}_0$$

- Answer: Use the eigenvalues of A:

Let λ_i be the eigenvalues of A,

and check stability for each λ_i.

- Stable if all λ_i satisfy stability conditions!

- Unstable if any λ_i violates stability conditions!

- Question: What about nonlinear systems of ODEs?

$$\frac{\mathrm{d}\vec{y}}{\mathrm{d}t} = f\left(t, \vec{y}\right) \qquad \text{with} \qquad \vec{y}\left(0\right) = \vec{y}_0$$

- Answer: Use the Jacobian of f:

$$\text{Let} \quad \lambda_i \text{ be the eigenvalues of } J_f := \left[\frac{\partial f_i}{\partial y_j}\right],$$

and check stability for each eigenvalueλ_i.