

Programming Languages and Compilers (CS 421)



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Based in part on slides by Mattox Beckman, as updated by Vikram Adve and Gul Agha



Axiomatic Semantics

- Also called Floyd-Hoare Logic
- Based on formal logic (first order predicate calculus)
- Axiomatic Semantics is a logical system built from *axioms* and *inference rules*
- Mainly suited to simple imperative programming languages



Axiomatic Semantics

- Used to formally prove a property (*post-condition*) of the *state* (the values of the program variables) after the execution of program, assuming another property (*pre-condition*) of the state holds before execution



Axiomatic Semantics

- Goal: Derive statements of form
$$\{P\} C \{Q\}$$
 - P , Q logical statements about state,
 P precondition, Q postcondition,
 C program
- Example: $\{x = 1\} x := x + 1 \{x = 2\}$



Axiomatic Semantics

- *Approach*: For each type of language statement, give an axiom or inference rule stating how to derive assertions of form

$$\{P\} C \{Q\}$$

where C is a statement of that type

- Compose axioms and inference rules to build proofs for complex programs



Axiomatic Semantics

- An expression $\{P\} C \{Q\}$ is a *partial correctness* statement
- For *total correctness* must also prove that C terminates (i.e. doesn't run forever)
 - Written: $[P] C [Q]$
- Will only consider partial correctness here



Language

- We will give rules for simple imperative language

<command>

::= <variable> := <term>

| <command>; ... ;<command>

| if <statement> then <command> else

<command>

| while <statement> do <command>

- Could add more features, like for-loops



Substitution

- Notation: $P[e/v]$ (sometimes $P[v \leftarrow e]$)
- Meaning: Replace every v in P by e
- Example:

$$(x + 2) [y-1/x] = ((y - 1) + 2)$$



The Assignment Rule

$$\frac{}{\{P [e/x]\} x := e \{P\}}$$

Example:

$$\frac{}{\{ \quad ? \quad \} x := y \{x = 2\}}$$



The Assignment Rule

$$\frac{}{\{P [e/x]\} x := e \{P\}}$$

Example:

$$\frac{}{\{_\ = 2\} x := y \{x = 2\}}$$



The Assignment Rule

$$\frac{}{\{P [e/x]\} x := e \{P\}}$$

Example:

$$\frac{}{\{y = 2\} x := y \{x = 2\}}$$



The Assignment Rule

$$\frac{}{\{P [e/x]\} x := e \{P\}}$$

Examples:

$$\frac{}{\{y = 2\} x := y \{x = 2\}}$$

$$\frac{}{\{y = 2\} x := 2 \{y = x\}}$$

$$\frac{}{\{x + 1 = n + 1\} x := x + 1 \{x = n + 1\}}$$

$$\frac{}{\{2 = 2\} x := 2 \{x = 2\}}$$



The Assignment Rule – Your Turn

- What is the weakest precondition of

$$x := x + y \{x + y = w - x\}?$$

$$\{ \quad ? \quad \}$$

$$x := x + y$$

$$\{x + y = w - x\}$$



The Assignment Rule – Your Turn

- What is the weakest precondition of

$$x := x + y \{x + y = w - x\}?$$

$$\{(x + y) + y = w - (x + y)\}$$

$$x := x + y$$

$$\{x + y = w - x\}$$



Precondition Strengthening

$$\frac{P \rightarrow P' \quad \{P'\} C \{Q\}}{\{P\} C \{Q\}}$$

- Meaning: If we can show that P implies P' ($P \rightarrow P'$) and we can show that $\{P'\} C \{Q\}$, then we know that $\{P\} C \{Q\}$
- P is *stronger* than P' means $P \rightarrow P'$



Precondition Strengthening

- Examples:

$$\frac{x = 3 \rightarrow x < 7 \quad \{x < 7\} x := x + 3 \quad \{x < 10\}}{\{x = 3\} x := x + 3 \quad \{x < 10\}}$$

$$\frac{\text{True} \rightarrow 2 = 2 \quad \{2 = 2\} x := 2 \quad \{x = 2\}}{\{\text{True}\} x := 2 \quad \{x = 2\}}$$

$$\frac{x = n \rightarrow x + 1 = n + 1 \quad \{x + 1 = n + 1\} x := x + 1 \quad \{x = n + 1\}}{\{x = n\} x := x + 1 \quad \{x = n + 1\}}$$



Which Inferences Are Correct?

$$\frac{\{x > 0 \ \& \ x < 5\} \ x := x * x \ \{x < 25\}}{\{x = 3\} \ x := x * x \ \{x < 25\}}$$

$$\frac{\{x = 3\} \ x := x * x \ \{x < 25\}}{\{x > 0 \ \& \ x < 5\} \ x := x * x \ \{x < 25\}}$$

$$\frac{\{x * x < 25\} \ x := x * x \ \{x < 25\}}{\{x > 0 \ \& \ x < 5\} \ x := x * x \ \{x < 25\}}$$

Which Inferences Are Correct?

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~~$$\frac{\{x = 3\} \ x := x * x \ \{x < 25\}}{\{x > 0 \ \& \ x < 5\} \ x := x * x \ \{x < 25\}}$$~~

$$\frac{\{x * x < 25\} \ x := x * x \ \{x < 25\}}{\{x > 0 \ \& \ x < 5\} \ x := x * x \ \{x < 25\}}$$



Sequencing

$$\frac{\{P\} C_1 \{Q\} \quad \{Q\} C_2 \{R\}}{\{P\} C_1; C_2 \{R\}}$$

- Example:

$$\frac{\begin{array}{l} \{z = z \ \& \ z = z\} \ x := z \ \{x = z \ \& \ z = z\} \\ \{x = z \ \& \ z = z\} \ y := z \ \{x = z \ \& \ y = z\} \end{array}}{\{z = z \ \& \ z = z\} \ x := z; \ y := z \ \{x = z \ \& \ y = z\}}$$

Sequencing

$$\frac{\{P\} C_1 \{Q\} \quad \{Q\} C_2 \{R\}}{\{P\} C_1; C_2 \{R\}}$$

■ Example:

$$\frac{\begin{array}{l} \{z = z \ \& \ z = z\} \ x := z \ \{x = z \ \& \ z = z\} \\ \{x = z \ \& \ z = z\} \ y := z \ \{x = z \ \& \ y = z\} \end{array}}{\{z = z \ \& \ z = z\} \ x := z; \ y := z \ \{x = z \ \& \ y = z\}}$$



Postcondition Weakening

$$\frac{\{P\} C \{Q'\} \quad Q' \rightarrow Q}{\{P\} C \{Q\}}$$

Example:

$$\frac{\{z = z \ \& \ z = z\} \ x := z; \ y := z \ \{x = z \ \& \ y = z\} \quad (x = z \ \& \ y = z) \rightarrow (x = y)}{\{z = z \ \& \ z = z\} \ x := z; \ y := z \ \{x = y\}}$$



Rule of Consequence

$$\frac{P \rightarrow P' \quad \{P'\} C \{Q'\} \quad Q' \rightarrow Q}{\{P\} C \{Q\}}$$

- Logically equivalent to the combination of Precondition Strengthening and Postcondition Weakening
- Uses $P \rightarrow P$ and $Q \rightarrow Q$



If Then Else

$$\frac{\{P \text{ and } B\} C_1 \{Q\} \quad \{P \text{ and (not } B)\} C_2 \{Q\}}{\{P\} \text{ if } B \text{ then } C_1 \text{ else } C_2 \{Q\}}$$

- Example: Want

$$\{y=a\}$$

if $x < 0$ then $y := y - x$ else $y := y + x$

$$\{y=a+|x|\}$$

Suffices to show:

(1) $\{y=a \ \& \ x < 0\} \ y := y - x \ \{y=a+|x|\}$ and

(4) $\{y=a \ \& \ \text{not}(x < 0)\} \ y := y + x \ \{y=a+|x|\}$


$$\{y=a \& x < 0\} \quad y := y - x \quad \{y = a + |x|\}$$

$$(3) \quad (y = a \& x < 0) \rightarrow y - x = a + |x|$$

$$(2) \quad \frac{\{y - x = a + |x|\} \quad y := y - x \quad \{y = a + |x|\}}{\quad}$$

$$(1) \quad \{y = a \& x < 0\} \quad y := y - x \quad \{y = a + |x|\}$$

(1) Reduces to (2) and (3) by
Precondition Strengthening

(2) Follows from assignment axiom

(3) Because $x < 0 \rightarrow |x| = -x$


$$\{y=a \wedge \text{not}(x < 0)\} \ y := y + x \ \{y = a + |x|\}$$

(6) $(y = a \wedge \text{not}(x < 0)) \rightarrow (y + x = a + |x|)$

(5) $\frac{\{y + x = a + |x|\} \ y := y + x \ \{y = a + |x|\}}{\quad}$

(4) $\{y = a \wedge \text{not}(x < 0)\} \ y := y + x \ \{y = a + |x|\}$

(4) Reduces to (5) and (6) by
Precondition Strengthening

(5) Follows from assignment axiom

(6) Because $\text{not}(x < 0) \rightarrow |x| = x$



If then else

$$\begin{array}{l} (1) \quad \{y=a \wedge x < 0\} y := y - x \{y = a + |x|\} \\ (4) \quad \{y = a \wedge \neg(x < 0)\} y := y + x \{y = a + |x|\} \\ \hline \{y = a\} \end{array}$$

if $x < 0$ then $y := y - x$ else $y := y + x$
 $\{y = a + |x|\}$

By the if_then_else rule



While

- We need a rule to be able to make assertions about **while** loops.
 - Inference rule because we can only draw conclusions if we know something about the body
 - Let's start with:

$$\frac{\{ \ ? \ } \ C \ \{ \ ? \ }}{\{ \ ? \ } \ \mathbf{while} \ B \ \mathbf{do} \ C \ \{ \ P \ }}$$



While

- The loop may never be executed, so if we want P to hold after, it had better hold before, so let's try:

$$\frac{\{ ? \} \quad C \quad \{ ? \}}{\{ P \} \quad \mathbf{while} \quad B \quad \mathbf{do} \quad C \quad \{ P \}}$$



While

- If all we know is P when we enter the **while** loop, then we all we know when we enter the body is $(P \text{ and } B)$
- If we need to know P when we finish the **while** loop, we had better know it when we finish the loop body:

$$\frac{\{ P \text{ and } B \} \ C \ \{ P \}}{\{ P \} \ \mathbf{while} \ B \ \mathbf{do} \ C \ \{ P \}}$$



While

- We can strengthen the previous rule because we also know that when the loop is finished, **not B** also holds
- Final **while** rule:

$$\frac{\{ P \text{ and } B \} C \{ P \}}{\{ P \} \text{ while } B \text{ do } C \{ P \text{ and not } B \}}$$



While

$$\frac{\{ P \text{ and } B \} \ C \ \{ P \}}{\{ P \} \ \mathbf{while} \ B \ \mathbf{do} \ C \ \{ P \text{ and not } B \}}$$

- P satisfying this rule is called a *loop invariant* because it must hold before and after the each iteration of the loop



While

- **While** rule generally needs to be used together with precondition strengthening and postcondition weakening
- There is NO algorithm for computing the correct P ; it requires intuition and an understanding of why the program works



Example

- Let us prove

$\{x \geq 0 \text{ and } x = a\}$

fact := 1;

while $x > 0$ do (fact := fact * x; x := x - 1)

$\{\text{fact} = a!\}$



Example

- We need to find a condition P that is true both before and after the loop is executed, and such that

$$(P \text{ and not } x > 0) \Rightarrow (\text{fact} = a!)$$



Example

- First attempt:

$$\{a! = \text{fact} * (x!)\}$$

- Motivation:
- What we want to compute: **a!**
- What we have computed: **fact**
which is the sequential product of **a** down through **(x + 1)**
- What we still need to compute: **x!**



Example

By post-condition weakening suffices to show

1. $\{x \geq 0 \text{ and } x = a\}$

fact := 1;

while $x > 0$ do (fact := fact * x; $x := x - 1$)

$\{a! = \text{fact} * (x!) \text{ and not } (x > 0)\}$

and

2. $\{a! = \text{fact} * (x!) \text{ and not } (x > 0)\} \rightarrow$
 $\{\text{fact} = a!\}$



Problem

2. $\{a! = \text{fact} * (x!) \text{ and not } (x > 0)\} \rightarrow \{\text{fact} = a!\}$
- Don't know this if $x < 0$
 - Need to know that $x = 0$ when loop terminates
 - Need a new loop invariant
 - Try adding $x \geq 0$
 - Then will have $x = 0$ when loop is done



Example

Second try, combine the two:

$$P = \{a! = \text{fact} * (x!) \text{ and } x \geq 0\}$$

Again, suffices to show

1. $\{x \geq 0 \text{ and } x = a\}$

fact := 1;

while $x > 0$ do (fact := fact * x; x := x - 1)

$\{P \text{ and not } x > 0\}$

and

2. $\{P \text{ and not } x > 0\} \rightarrow \{\text{fact} = a!\}$



Example

- For 2, we need

$\{a! = \text{fact} * (x!) \text{ and } x \geq 0 \text{ and not } (x > 0)\} \rightarrow$
 $\{\text{fact} = a!\}$

But $\{x \geq 0 \text{ and not } (x > 0)\} \rightarrow \{x = 0\}$ so
 $\text{fact} * (x!) = \text{fact} * (0!) = \text{fact}$

Therefore

$\{a! = \text{fact} * (x!) \text{ and } x \geq 0 \text{ and not } (x > 0)\} \rightarrow$
 $\{\text{fact} = a!\}$



Example

- For 1, by the sequencing rule it suffices to show

3. $\{x \geq 0 \text{ and } x = a\}$
 $\text{fact} := 1$
 $\{a! = \text{fact} * (x!) \text{ and } x \geq 0\}$

And

4. $\{a! = \text{fact} * (x!) \text{ and } x \geq 0\}$
 while $x > 0$ do
 $(\text{fact} := \text{fact} * x; x := x - 1)$
 $\{a! = \text{fact} * (x!) \text{ and } x \geq 0 \text{ and not } (x > 0)\}$



Example

- Suffices to show that

$$\{a! = \text{fact} * (x!) \text{ and } x \geq 0\}$$

holds before the while loop is entered and that if

$$\{(a! = \text{fact} * (x!)) \text{ and } x \geq 0 \text{ and } x > 0\}$$

holds before we execute the body of the loop, then

$$\{(a! = \text{fact} * (x!)) \text{ and } x \geq 0\}$$

holds after we execute the body



Example

By the assignment rule, we have

$$\{a! = 1 * (x!) \text{ and } x \geq 0\}$$
$$\text{fact} := 1$$
$$\{a! = \text{fact} * (x!) \text{ and } x \geq 0\}$$

Therefore, to show (3), by precondition strengthening, it suffices to show

$$(x \geq 0 \text{ and } x = a) \rightarrow \\ (a! = 1 * (x!) \text{ and } x \geq 0)$$



Example

$(x \geq 0 \text{ and } x = a) \rightarrow$

$(a! = 1 * (x!) \text{ and } x \geq 0)$

holds because $x = a \rightarrow x! = a!$

Have that $\{a! = \text{fact} * (x!) \text{ and } x \geq 0\}$

holds at the start of the while loop



Example

To show (4):

$\{a! = \text{fact} * (x!) \text{ and } x \geq 0\}$

while $x > 0$ do

($\text{fact} := \text{fact} * x; x := x - 1$)

$\{a! = \text{fact} * (x!) \text{ and } x \geq 0 \text{ and not } (x > 0)\}$

we need to show that

$\{(a! = \text{fact} * (x!)) \text{ and } x \geq 0\}$

is a loop invariant



Example

We need to show:

$$\{(a! = \text{fact} * (x!)) \text{ and } x \geq 0 \text{ and } x > 0\}$$
$$(\text{fact} = \text{fact} * x; x := x - 1)$$
$$\{(a! = \text{fact} * (x!)) \text{ and } x \geq 0\}$$

We will use assignment rule,
sequencing rule and precondition
strengthening



Example

By the assignment rule, we have

$$\{(a! = \text{fact} * ((x-1)!)) \text{ and } x - 1 \geq 0\}$$

$$x := x - 1$$

$$\{(a! = \text{fact} * (x!)) \text{ and } x \geq 0\}$$

By the sequencing rule, it suffices to show

$$\{(a! = \text{fact} * (x!)) \text{ and } x \geq 0 \text{ and } x > 0\}$$

$$\text{fact} = \text{fact} * x$$

$$\{(a! = \text{fact} * ((x-1)!)) \text{ and } x - 1 \geq 0\}$$



Example

By the assignment rule, we have that

$$\{(a! = (\text{fact} * x) * ((x-1)!)) \text{ and } x - 1 \geq 0\}$$
$$\text{fact} = \text{fact} * x$$

$$\{(a! = \text{fact} * ((x-1)!)) \text{ and } x - 1 \geq 0\}$$

By Precondition strengthening, it suffices to show that

$$((a! = \text{fact} * (x!)) \text{ and } x \geq 0 \text{ and } x > 0) \rightarrow$$

$$((a! = (\text{fact} * x) * ((x-1)!)) \text{ and } x - 1 \geq 0)$$



Example

However

$$\text{fact} * x * (x - 1)! = \text{fact} * x$$

and $(x > 0) \rightarrow x - 1 \geq 0$

since x is an integer, so

$$\{(a! = \text{fact} * (x!)) \text{ and } x \geq 0 \text{ and } x > 0\} \rightarrow$$

$$\{(a! = (\text{fact} * x) * ((x-1)!)) \text{ and } x - 1 \geq 0\}$$



Example

Therefore, by precondition strengthening

$$\{(a! = \text{fact} * (x!)) \text{ and } x \geq 0 \text{ and } x > 0\}$$
$$\text{fact} = \text{fact} * x$$
$$\{(a! = \text{fact} * ((x-1)!)) \text{ and } x - 1 \geq 0\}$$

This finishes the proof