Programming Languages and Compilers (CS 421)

Elsa L Gunter
2112 SC, UIUC
http://courses.engr.illinois.edu/cs421

Based in part on slides by Mattox Beckman, as updated by Vikram Adve and Gul Agha

Example Evaluation

- First step:

$$\frac{}{\text{(if x > 5 then y:= 2 + 3 else y:=3 + 4 fi, \{x -> 7\}) --> ?}}$$

Example Evaluation

- First step:

$$\frac{\text{(x > 5, \{x -> 7\}) --> ?}}{\text{(if x > 5 then y:= 2 + 3 else y:=3 + 4 fi, \{x -> 7\}) --> ?}}$$

Example Evaluation

- First step:

$$\frac{\text{(x,\{x -> 7\}) --> (7, \{x -> 7\})}}{\frac{\text{(x > 5, \{x -> 7\}) --> ?}}{\text{(if x > 5 then y:= 2 + 3 else y:=3 + 4 fi, \{x -> 7\}) --> ?}}}$$

Example Evaluation

- First step:

$$\frac{\text{(x,\{x -> 7\}) --> (7, \{x -> 7\})}}{\frac{\text{(x > 5, \{x -> 7\}) --> (7 > 5, \{x -> 7\})}}{\text{(if x > 5 then y:= 2 + 3 else y:=3 + 4 fi, \{x -> 7\}) --> ?}}}$$

Example Evaluation

- First step:

$$\frac{\text{(x,\{x -> 7\}) --> (7, \{x -> 7\})}}{\frac{\text{(x > 5, \{x -> 7\}) --> (7 > 5, \{x -> 7\})}}{\text{(if x > 5 then y:= 2 + 3 else y:=3 + 4 fi, \{x -> 7\})}}}$$

--> (if 7 > 5 then y:=2 + 3 else y:=3 + 4 fi, {x -> 7})

Example Evaluation

- Second Step:

$$\frac{(7 > 5, \{x \to 7\}) \to (\text{true}, \{x \to 7\})}{\begin{array}{c}(\text{if } 7 > 5 \text{ then } y:=2+3 \text{ else } y:=3+4 \text{ fi}, \{x \to 7\}) \\ \to (\text{if true then } y:=2+3 \text{ else } y:=3+4 \text{ fi}, \{x \to 7\})\end{array}}$$

- Third Step:

(if true then y:=2 + 3 else y:=3 + 4 fi, {x -> 7})
-->(y:=2+3, {x->7})

Example Evaluation

- Fourth Step:

$$\frac{(2+3, \{x \to 7\}) \to (5, \{x \to 7\})}{(y:=2+3, \{x \to 7\}) \to (y:=5, \{x \to 7\})}$$

- Fifth Step:

(y:=5, {x->7}) --> {y -> 5, x -> 7}

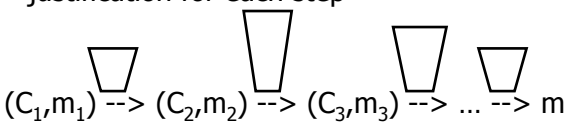
Example Evaluation

- Bottom Line:

(if x > 5 then y:= 2 + 3 else y:=3 + 4 fi, {x -> 7})
--> (if 7 > 5 then y:=2 + 3 else y:=3 + 4 fi, {x -> 7})
-->(if true then y:=2 + 3 else y:=3 + 4 fi, {x -> 7})
-->(y:=2+3, {x->7})
--> (y:=5, {x->7}) --> {y -> 5, x -> 7}

Transition Semantics Evaluation

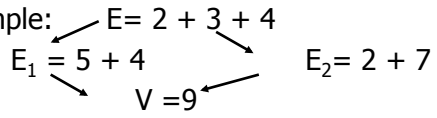
- A sequence of steps with trees of justification for each step



- Let -->* be the transitive closure of -->
- Ie, the smallest transitive relation containing -->

Church-Rosser Property

- Church-Rosser Property: If E-->* E_1 and E-->* E_2, if there exists a value V such that E_1 -->* V, then E_2 -->* V
- Also called **confluence** or **diamond property**
- Example:



Does It always Hold?

- No. Languages with side-effects tend not be Church-Rosser with the combination of call-by-name and call-by-value
- Alonzo Church and Barkley Rosser proved in 1936 the λ-calculus does have it
- Benefit of Church-Rosser: can check equality of terms by evaluating them (Given evaluation strategy might not terminate, though)

Lambda Calculus - Motivation

- Aim is to capture the essence of functions, function applications, and evaluation
- λ–calculus is a theory of computation
- "The Lambda Calculus: Its Syntax and Semantics". H. P. Barendregt. North Holland, 1984

Lambda Calculus - Motivation

- All *sequential programs* may be viewed as functions from input (initial state and input values) to output (resulting state and output values).
- λ-calculus is a mathematical formalism of functions and functional computations
- Two flavors: typed and untyped

Untyped λ-Calculus

- Only three kinds of expressions:
  - Variables: x, y, z, w, …
  - Abstraction: λ x. e
    (Function creation, think fun x -> e)
  - Application: e_1 e_2

Untyped λ-Calculus Grammar

- Formal BNF Grammar:
  - <expression> ::= <variable>
    | <abstraction>
    | <application>
    | (<expression>)
  - <abstraction>
    ::= λ<variable>.<expression>
  - <application>
    ::= <expression> <expression>

Untyped λ-Calculus Terminology

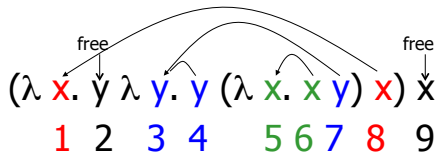
- Occurrence: a location of a subterm in a term
- Variable binding: λ x. e is a binding of x in e
- Bound occurrence: all occurrences of x in λ x. e
- Free occurrence: one that is not bound
- Scope of binding: in λ x. e, all occurrences in e not in a subterm of the form λ x. e' (same x)
- Free variables: all variables having free occurrences in a term

Example

- Label occurrences and scope:

(λ x. y λ y. y (λ x. x y) x) x
1 2 3 4 5 6 7 8 9

Example

- Label occurrences and scope:
- 

Untyped λ-Calculus

- How do you compute with the λ-calculus?
- Roughly speaking, by substitution:
- $(\lambda\ x.\ e_1)\ e_2 \Rightarrow^* e_1\,[e_2 / x]$
- * Modulo all kinds of subtleties to avoid free variable capture

Transition Semantics for λ-Calculus

$$\frac{E \text{ --> } E''}{E\ E' \text{ --> } E''\ E'}$$

- Application (version 1 - Lazy Evaluation)

$$(\lambda\ x\ .\ E)\ E' \text{ --> } E[E'/x]$$

- Application (version 2 - Eager Evaluation)

$$\frac{E' \text{ --> } E''}{(\lambda\ x\ .\ E)\ E' \text{ --> } (\lambda\ x\ .\ E)\ E''}$$

$$\frac{}{(\lambda\ x\ .\ E)\ V \text{ --> } E[V/x]}$$

V - variable or abstraction (value)

How Powerful is the Untyped λ-Calculus?

- The untyped λ-calculus is Turing Complete
  - Can express any sequential computation
- Problems:
  - How to express basic data: booleans, integers, etc?
  - How to express recursion?
  - Constants, if_then_else, etc, are conveniences; can be added as syntactic sugar

Typed vs Untyped λ-Calculus

- The *pure* λ-calculus has no notion of type: (f f) is a legal expression
- Types restrict which applications are valid
- Types are not syntactic sugar! They disallow some terms
- Simply typed λ-calculus is less powerful than the untyped λ-Calculus: NOT Turing Complete (no recursion)

Uses of λ-Calculus

- Typed and untyped λ-calculus used for theoretical study of sequential programming languages
- Sequential programming languages are essentially the λ-calculus, extended with predefined constructs, constants, types, and syntactic sugar
- Ocaml is close to the λ-Calculus:

fun x -> exp --> λ x. exp

let x = e_1 in e_2 --> $(\lambda\ x.\ e_2)e_1$

α Conversion

- α-conversion:
  λ x. exp --α--> λ y. (exp [y/x])
- Provided that
  1. y is not free in exp
  2. No free occurrence of x in exp becomes bound in exp when replaced by y

α Conversion Non-Examples

1. Error: y is not free in termsecond
   λ x. x y --α--> λ y. y y
2. Error: free occurrence of x becomes bound in wrong way when replaced by y
   λ x. λ y. x y --α--> λ y. λ y. y y
   exp exp[y/x]

But λ x. (λ y. y) x --α--> λ y. (λ y. y) y
And λ y. (λ y. y) y --α--> λ x. (λ y. y) x

Congruence

- Let ~ be a relation on lambda terms. ~ is a congruence if
- it is an equivalence relation
- If $e_1 \sim e_2$ then
  - $(e\ e_1) \sim (e\ e_2)$ and $(e_1 e) \sim (e_2\ e)$
  - $\lambda x.\ e_1 \sim \lambda x.\ e_2$

α Equivalence

- α equivalence is the smallest congruence containing α conversion
- One usually treats α-equivalent terms as equal - i.e. use α equivalence classes of terms

Example

Show: λ x. (λ y. y x) x ~α~ λ y. (λ x. x y) y

- λ x. (λ y. y x) x --α--> λ z. (λ y. y z) z so
  λ x. (λ y. y x) x ~α~ λ z. (λ y. y z) z
- (λ y. y z) --α--> (λ x. x z) so
  (λ y. y z) ~α~ (λ x. x z) so
  λ z. (λ y. y z) z ~α~ λ z. (λ x. x z) z
- λ z. (λ x. x z) z --α--> λ y. (λ x. x y) y so
  λ z. (λ x. x z) z ~α~ λ y. (λ x. x y) y
- λ x. (λ y. y x) x ~α~ λ y. (λ x. x y) y

Substitution

- Defined on α-equivalence classes of terms
- P [N / x] means replace every free occurrence of x in P by N
  - P called *redex*; N called *residue*
- Provided that no variable free in P becomes bound in P [N / x]
  - Rename bound variables in P to avoid capturing free variables of N

Substitution

- x [N / x] = N
- y [N / x] = y if y ≠ x
- (e_1 e_2) [N / x] = ((e_1 [N / x]) (e_2 [N / x]))
- (λ x. e) [N / x] = (λ x. e)
- (λ y. e) [N / x] = λ y. (e [N / x]) provided y ≠ x and y not free in N
  - Rename y in redex if necessary

Example

(λ y. y z) [(λ x. x y) / z] = ?

- Problems?
  - z in redex in scope of y binding
  - y free in the residue
- (λ y. y z) [(λ x. x y) / z] --α--> (λ w.w z) [(λ x. x y) / z] = λ w. w (λ x. x y)

Example

- Only replace free occurrences
- (λ y. y z (λ z. z)) [(λ x. x) / z] = λ y. y (λ x. x) (λ z. z)

Not

λ y. y (λ x. x) (λ z. (λ x. x))

β reduction

- β Rule: (λ x. P) N --β--> P [N /x]
- Essence of computation in the lambda calculus
- Usually defined on α-equivalence classes of terms

Example

- (λ z. (λ x. x y) z) (λ y. y z)
  --β--> (λ x. x y) (λ y. y z)
  --β--> (λ y. y z) y --β--> y z
- (λ x. x x) (λ x. x x)
  --β--> (λ x. x x) (λ x. x x)
  --β--> (λ x. x x) (λ x. x x) --β--> ….

α β Equivalence

- α β equivalence is the smallest congruence containing α equivalence and β reduction
- A term is in *normal form* if no subterm is α equivalent to a term that can be β reduced
- Hard fact (Church-Rosser): if e_1 and e_2 are αβ-equivalent and both are normal forms, then they are α equivalent

Order of Evaluation

- Not all terms reduce to normal forms
- Not all reduction strategies will produce a normal form if one exists

Lazy evaluation:

- Always reduce the left-most application in a top-most series of applications (i.e. Do not perform reduction inside an abstraction)
- Stop when term is not an application, or left-most application is not an application of an abstraction to a term

Example 1

- (λ z. (λ x. x)) ((λ y. y y) (λ y. y y))
- Lazy evaluation:
- Reduce the left-most application:
- (λ z. (λ x. x)) ((λ y. y y) (λ y. y y)) --β--> (λ x. x)

Eager evaluation

- (Eagerly) reduce left of top application to an abstraction
- Then (eagerly) reduce argument
- Then β-reduce the application

Example 1

- (λ z. (λ x. x))((λ y. y y) (λ y. y y))
- Eager evaluation:
- Reduce the rator of the top-most application to an abstraction: Done.
- Reduce the argument:
- (λ z. (λ x. x))((λ y. y y) (λ y. y y))

--β--> (λ z. (λ x. x))((λ y. y y) (λ y. y y))

--β--> (λ z. (λ x. x))((λ y. y y) (λ y. y y))…

Example 2

- (λ x. x x)((λ y. y y) (λ z. z))
- Lazy evaluation:

(λ x. x x)((λ y. y y) (λ z. z)) --β-->

Example 2

- (λ x. x x)((λ y. y y) (λ z. z))
- Lazy evaluation:

(λ x. x x)((λ y. y y) (λ z. z)) --β-->

Example 2

- (λ x. x x)((λ y. y y) (λ z. z))
- Lazy evaluation:

(λ x. x x)((λ y. y y) (λ z. z)) --β-->

((λ y. y y) (λ z. z)) ((λ y. y y) (λ z. z))

Example 2

- (λ x. x x)((λ y. y y) (λ z. z))
- Lazy evaluation:

(λ x. x x)((λ y. y y) (λ z. z)) --β-->

((λ y. y y) (λ z. z)) ((λ y. y y) (λ z. z)

Example 2

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Example 2

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- Lazy evaluation:

(λ x. x x)((λ y. y y) (λ z. z)) --β-->

((λ y. y y) (λ z. z)) ((λ y. y y) (λ z. z))

--β--> ((λ z. z) (λ z. z))((λ y. y y) (λ z. z))

Example 2

- (λ x. x x)((λ y. y y) (λ z. z))
- Lazy evaluation:

(λ x. x x)((λ y. y y) (λ z. z)) --β-->

((λ y. y y) (λ z. z)) ((λ y. y y) (λ z. z))

--β--> ((λ z. z) (λ z. z))(λ y. y y) (λ z. z))

Example 2

- (λ x. x x)((λ y. y y) (λ z. z))
- Lazy evaluation:

(λ x. x x)((λ y. y y) (λ z. z)) --β-->
((λ y. y y) (λ z. z)) ((λ y. y y) (λ z. z))
--β--> ((λ z. z) (λ z. z))((λ y. y y) (λ z. z))

Example 2

- (λ x. x x)((λ y. y y) (λ z. z))
- Lazy evaluation:

(λ x. x x)((λ y. y y) (λ z. z)) --β-->
((λ y. y y) (λ z. z)) ((λ y. y y) (λ z. z))
--β--> ((λ z. z) (λ z. z))((λ y. y y) (λ z. z))
--β--> (λ z. z) ((λ y. y y) (λ z. z))

Example 2

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((λ y. y y) (λ z. z)) ((λ y. y y) (λ z. z))
--β--> ((λ z. z) (λ z. z))((λ y. y y) (λ z. z))
--β--> (λ z. z) ((λ y. y y) (λ z. z)) --β-->
(λ y. y y) (λ z. z)

Example 2

- (λ x. x x)((λ y. y y) (λ z. z))
- Lazy evaluation:

(λ x. x x)((λ y. y y) (λ z. z)) --β-->
((λ y. y y) (λ z. z)) ((λ y. y y) (λ z. z))
--β--> ((λ z. z) (λ z. z))((λ y. y y) (λ z. z))
--β--> (λ z. z) ((λ y. y y) (λ z. z)) --β-->
(λ y. y y) (λ z. z)

Example 2

- (λ x. x x)((λ y. y y) (λ z. z))
- Lazy evaluation:

(λ x. x x)((λ y. y y) (λ z. z)) --β-->
((λ y. y y) (λ z. z)) ((λ y. y y) (λ z. z))
--β--> ((λ z. z) (λ z. z))((λ y. y y) (λ z. z))
--β--> (λ z. z) ((λ y. y y) (λ z. z)) --β-->
(λ y. y y) (λ z. z) ~β~ λ z. z

Example 2

- (λ x. x x)((λ y. y y) (λ z. z))
- Eager evaluation:

(λ x. x x) ((λ y. y y) (λ z. z)) --β-->
(λ x. x x) ((λ z. z) (λ z. z)) --β-->
(λ x. x x) (λ z. z) --β-->
(λ z. z) (λ z. z) --β--> λ z. z