Programming Languages and Compilers (CS 421)

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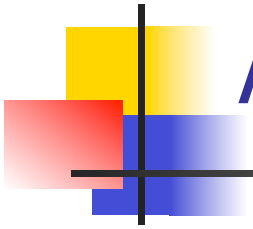
Based in part on slides by Mattox Beckman, as updated by Vikram Adve and Gul Agha

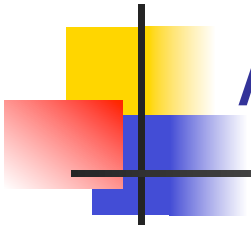


Axioms - Constants

$$\frac{}{\Gamma \vdash n : \text{int}}$$ (assuming n is an integer constant)

$$\frac{}{\Gamma \vdash \text{true} : \text{bool}} \qquad \frac{}{\Gamma \vdash \text{false} : \text{bool}}$$

- These rules are true with any typing environment
- Γ, n are meta-variables



Axioms – Variables (Monomorphic Rule)

Notation: Let $\Gamma(x) = \sigma$ if $x : \sigma \in \Gamma$

Note: if such σ exits, its unique

Variable axiom:

$$\frac{}{\Gamma \mid\!- x : \sigma} \quad \text{if } \Gamma(x) = \sigma$$

Simple Rules - Arithmetic

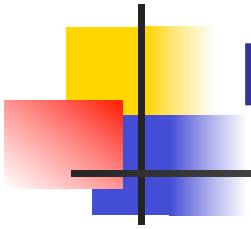
Primitive operators ($\oplus \in \{ +, -, *, \ldots \}$):

$$\frac{\Gamma \mid\!- e_1 : \tau_1 \quad \Gamma \mid\!- e_2 : \tau_2 \quad (\oplus) : \tau_1 \rightarrow \tau_2 \rightarrow \tau_3}{\Gamma \mid\!- e_1 \oplus e_2 : \tau_3}$$

Relations ($\sim \in \{ <, >, =, <=, >= \}$):

$$\frac{\Gamma \mid\!- e_1 : \tau \quad \Gamma \mid\!- e_2 : \tau}{\Gamma \mid\!- e_1 \sim e_2 : \text{bool}}$$

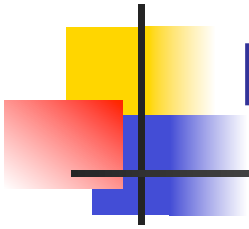
For the moment, think τ is int



Example: {x:int} |- x + 2 = 3 :bool

What do we need to show first?

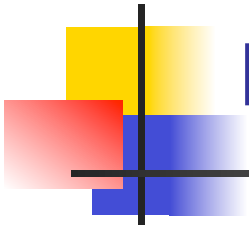
{x:int} |- x + 2 = 3 : bool



Example: {x:int} |- x + 2 = 3 :bool

What do we need for the left side?

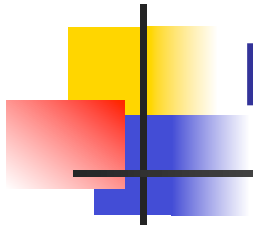
$$\frac{\{x : int\} \vdash x + 2 : int \qquad \{x:int\} \vdash 3 : int}{\{x:int\} \vdash x + 2 = 3 : bool}\ Rel$$



Example: {x:int} |- x + 2 = 3 :bool

How to finish?

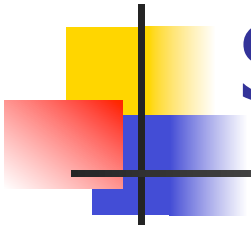
$$\dfrac{\dfrac{\{x{:}int\} \vdash x{:}int \quad \{x{:}int\} \vdash 2{:}int}{\{x : int\} \vdash x + 2 : int}\,AO \qquad \{x{:}int\} \vdash 3 : int}{\{x{:}int\} \vdash x + 2 = 3 : bool}\,Rel$$



Example: {x:int} |- x + 2 = 3 :bool

Complete Proof (type derivation)

$$\dfrac{\dfrac{\dfrac{}{\{x{:}int\} \vdash x{:}int}\,Var \quad \dfrac{}{\{x{:}int\} \vdash 2{:}int}\,Const}{\{x : int\} \vdash x + 2 : int}\,AO \quad \dfrac{}{\{x{:}int\} \vdash 3 :int}\,Const}{\{x{:}int\} \vdash x + 2 = 3 : bool}\,Rel$$

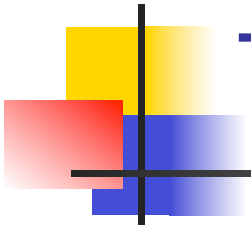


Simple Rules - Booleans

Connectives

$$\frac{\Gamma \vdash e_1 : \text{bool} \qquad \Gamma \vdash e_2 : \text{bool}}{\Gamma \vdash e_1 \;\&\&\; e_2 : \text{bool}}$$

$$\frac{\Gamma \vdash e_1 : \text{bool} \qquad \Gamma \vdash e_2 : \text{bool}}{\Gamma \vdash e_1 \;||\; e_2 : \text{bool}}$$

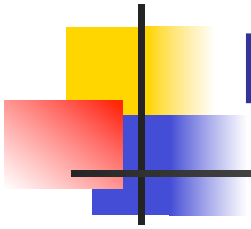


Type Variables in Rules

- If_then_else rule:

$$\frac{\Gamma \vdash e_1 : \text{bool} \quad \Gamma \vdash e_2 : \tau \quad \Gamma \vdash e_3 : \tau}{\Gamma \vdash (\text{if } e_1 \text{ then } e_2 \text{ else } e_3) : \tau}$$

- τ is a type variable (meta-variable)
- Can take any type at all
- All instances in a rule application must get same type
- Then branch, else branch and if_then_else must all have same type



Function Application

- Application rule:

$$\frac{\Gamma \mid\!- e_1 : \tau_1 \rightarrow \tau_2 \quad \Gamma \mid\!- e_2 : \tau_1}{\Gamma \mid\!- (e_1\ e_2) : \tau_2}$$

- If you have a function expression e_1 of type $\tau_1 \rightarrow \tau_2$ applied to an argument e_2 of type τ_1, the resulting expression $e_1 e_2$ has type τ_2

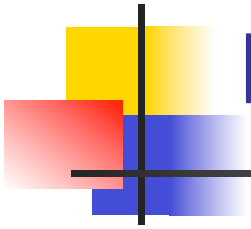
Application Examples

$$\frac{\Gamma \text{ |- print\_int : int} \rightarrow \text{unit} \quad \Gamma \text{ |- 5 : int}}{\Gamma \text{ |- (print\_int 5) : unit}}$$

- e_1 = print_int, e_2 = 5,
- τ_1 = int, τ_2 = unit

$$\frac{\Gamma \text{ |- map print\_int : int list} \rightarrow \text{unit list} \quad \Gamma \text{ |- [3;7] : int list}}{\Gamma \text{ |- (map print\_int [3; 7]) : unit list}}$$

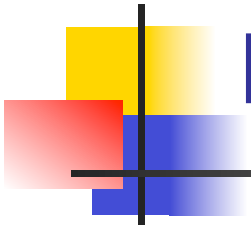
- e_1 = map print_int, e_2 = [3; 7],
- τ_1 = int list, τ_2 = unit list



Fun Rule

- Rules describe types, but also how the environment Γ may change
- Can only do what rule allows!
- fun rule:

$$\frac{\{x : \tau_1\} + \Gamma \mid\!- e : \tau_2}{\Gamma \mid\!- \text{fun } x \text{ -> } e : \tau_1 \rightarrow \tau_2}$$



Fun Examples

$$\frac{\{y : int\} + \Gamma \vdash y + 3 : int}{\Gamma \vdash fun\ y \rightarrow y + 3 : int \rightarrow int}$$

$$\frac{\{f : int \rightarrow bool\} + \Gamma \vdash f\ 2 :: [true] : bool\ list}{\Gamma \vdash (fun\ f \rightarrow f\ 2 :: [true]) : (int \rightarrow bool) \rightarrow bool\ list}$$

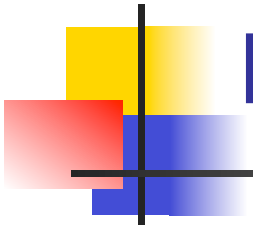
(Monomorphic) Let and Let Rec

- let rule:

$$\frac{\Gamma \vdash e_1 : \tau_1 \quad \{x : \tau_1\} + \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash (\text{let } x = e_1 \text{ in } e_2) : \tau_2}$$

- let rec rule:

$$\frac{\{x : \tau_1\} + \Gamma \vdash e_1 : \tau_1 \quad \{x : \tau_1\} + \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash (\text{let rec } x = e_1 \text{ in } e_2) : \tau_2}$$



Example

- Which rule do we apply?

$$\frac{?}{\begin{array}{l}\text{|- (let rec one = 1 :: one in}\\ \quad\text{let x = 2 in}\\ \quad\quad\text{fun y -> (x :: y :: one)) : int} \rightarrow \text{int list}\end{array}}$$

Example

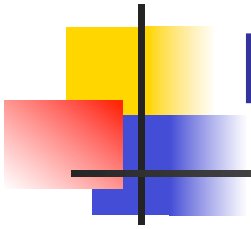
- Let rec rule:

① {one : int list} |- (1 :: one) : int list

② {one : int list} |- (let x = 2 in fun y -> (x :: y :: one)) : int → int list

LR

|- (let rec one = 1 :: one in
    let x = 2 in
      fun y -> (x :: y :: one)) : int → int list



Proof of 1

- Which rule?

{one : int list} |- (1 :: one) : int list

Proof of 1

- Primitive Operation

(::): int →int list→ int list

③ ④

{one : int list} |- 1:int

{one : int list} |- one : int list

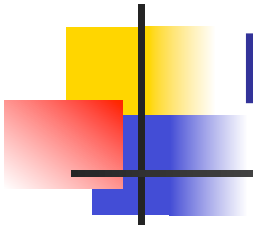
PO

{one : int list} |- (1 :: one) : int list

Proof of 1

(::): int →int list→ int list

$$\dfrac{\dfrac{\dfrac{}{\{one : int\ list\} \vdash 1:int}\ ③\ \text{Constant Rule} \qquad \dfrac{}{\{one : int\ list\} \vdash one : int\ list}\ ④\ \text{Variable Rule}}{\{one : int\ list\} \vdash (1 :: one) : int\ list}\ \text{PO}}$$



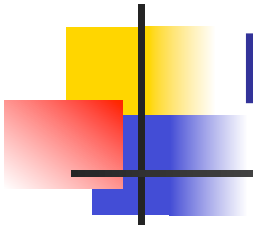
Proof of 2

?

{one : int list} |- (let x = 2 in
fun y -> (x :: y :: one)) : int → int list

Proof of 2

$$\frac{\dfrac{}{\{one : int\ list\} \vdash 2 : int}\ \text{Constant} \qquad \textcircled{5}\ \{x:int;\ one : int\ list\} \vdash \text{fun } y \rightarrow (x :: y :: one) : int \rightarrow int\ list}{\{one : int\ list\} \vdash (\text{let } x = 2 \text{ in } \text{fun } y \rightarrow (x :: y :: one)) : int \rightarrow int\ list}\ \text{Let}$$



Proof of 5

$$\frac{?}{\{x{:}\mathrm{int};\ \mathrm{one} : \mathrm{int\ list}\} \vdash \mathrm{fun}\ y \rightarrow (x :: y :: \mathrm{one})) : \mathrm{int} \rightarrow \mathrm{int\ list}}$$

Proof of 5

$$\dfrac{\dfrac{?}{\{y{:}int;\ x{:}int;\ one : int\ list\} \vdash (x :: y :: one) : int\ list}\ \text{Fun}}{\{x{:}int;\ one : int\ list\} \vdash fun\ y \rightarrow (x :: y :: one)) : int \rightarrow int\ list}$$

Proof of 5

- Primitive Operation

(::): int →int list→ int list

⑥ {y:int; x:int; one:int list} |- x:int list

⑦ {y:int; x:int; one:int list} |- (y :: one) : int list

PO

{y:int; x:int; one : int list} |- (x :: y :: one) : int list

Fun

{x:int; one : int list} |- fun y -> (x :: y :: one))
: int → int list

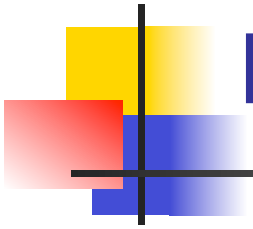
Proof of 5

- Primitive Operation

(::): int →int list→ int list

$$\dfrac{\dfrac{\textcircled{6}\ \text{Variable Rule} \qquad \textcircled{7}}{\{y{:}int;\ x{:}int;\ one{:}int\ list\} \vdash x{:}int\ list \quad \{y{:}int;\ x{:}int;\ one{:}int\ list\} \vdash (y :: one) : int\ list}\ \text{PO}}{\{y{:}int;\ x{:}int;\ one : int\ list\} \vdash (x :: y :: one) : int\ list}\ \text{Fun}$$

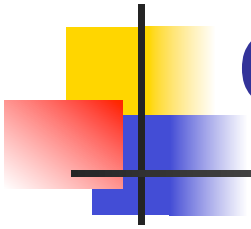
$$\{x{:}int;\ one : int\ list\} \vdash fun\ y \rightarrow (x :: y :: one)) : int \rightarrow int\ list$$



Proof of 7

(::): int →int list→ int list

$$\dfrac{\dfrac{}{\{y{:}int;\ x{:}int;\ one{:}int\ list\} \vdash x{:}int\ list}\ \text{Variable Rule} \qquad \dfrac{}{\{y{:}int;\ x{:}int;\ one{:}int\ list\} \vdash one : int\ list}\ \text{Variable Rule}}{\{y{:}int;\ x{:}int;\ one : int\ list\} \vdash (y :: one)) : int\ list}\ \text{PO}\ \text{PO}$$



Curry - Howard Isomorphism

- Type Systems are logics; logics are type systems
- Types are propositions; propositions are types
- Terms are proofs; proofs are terms

- Functions space arrow corresponds to implication; application corresponds to modus ponens

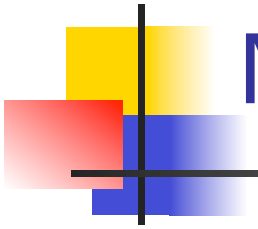
Curry - Howard Isomorphism

- Modus Ponens

$$\frac{A \Rightarrow B \quad A}{B}$$

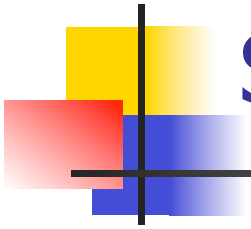
- Application

$$\frac{\Gamma \mid\!- e_1 : \alpha \rightarrow \beta \quad \Gamma \mid\!- e_2 : \alpha}{\Gamma \mid\!- (e_1\ e_2) : \beta}$$



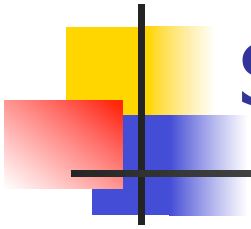
Mia Copa

- The above system can't handle polymorphism as in OCAML
- No type variables in type language (only meta-variable in the logic)
- Need:
  - Object level type variables and some kind of type quantification
  - **let** and **let rec** rules to introduce polymorphism
  - Explicit rule to eliminate (instantiate) polymorphism



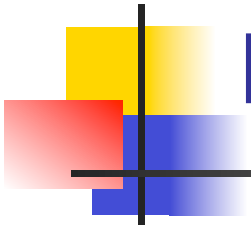
Support for Polymorphic Types

- Monomorpic Types (τ):
  - Basic Types: int, bool, float, string, unit, …
  - Type Variables: α, β, γ, δ, ε
  - Compound Types: $\alpha \rightarrow \beta$, int * string, bool list, …
- Polymorphic Types:
  - Monomorphic types τ
  - Universally quantified monomorphic types
  - $\forall \alpha_1, \ldots, \alpha_n \,.\, \tau$
  - Can think of τ as same as $\forall \,.\, \tau$



Support for Polymorphic Types

- Typing Environment Γ supplies polymorphic types (which will often just be monomorphic) for variables
- Free variables of monomorphic type just type variables that occur in it
  - Write FreeVars(τ)
- Free variables of polymorphic type removes variables that are universally quantified
  - FreeVars($\forall \alpha_1, \ldots, \alpha_n . \tau$) = FreeVars($\tau$) – $\{\alpha_1, \ldots, \alpha_n\}$
- FreeVars(Γ) = all FreeVars of types in range of Γ

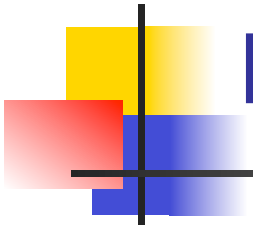


Monomorphic to Polymorphic

- Given:
  - type environment Γ
  - monomorphic type τ
  - τ shares type variables with Γ
- Want most polymorphic type for τ that doesn't break sharing type variables with Γ
- $\text{Gen}(\tau, \Gamma) = \forall \alpha_1, \ldots, \alpha_n . \tau$ where

  $\{\alpha_1, \ldots, \alpha_n\} = \text{freeVars}(\tau) - \text{freeVars}(\Gamma)$

Polymorphic Typing Rules



- A *type judgement* has the form

  Γ |- exp : τ

  - Γ uses polymorphic types
  - τ still monomorphic
- Most rules stay same (except use more general typing environments)
- Rules that change:
  - Variables
  - Let and Let Rec
  - Allow polymorphic constants
- Worth noting functions again

Polymorphic Let and Let Rec

- let rule:

$$\frac{\Gamma \mid\!- e_1 : \tau_1 \quad \{x : \mathrm{Gen}(\tau_1,\Gamma)\} + \Gamma \mid\!- e_2 : \tau_2}{\Gamma \mid\!- (\text{let } x = e_1 \text{ in } e_2) : \tau_2}$$

- let rec rule:

$$\frac{\{x : \tau_1\} + \Gamma \mid\!- e_1 : \tau_1 \quad \{x : \mathrm{Gen}(\tau_1,\Gamma)\} + \Gamma \mid\!- e_2 : \tau_2}{\Gamma \mid\!- (\text{let rec } x = e_1 \text{ in } e_2) : \tau_2}$$

Polymorphic Variables (Identifiers)

Variable axiom:

$$\frac{}{\Gamma \mid\!- x : \varphi(\tau)} \quad \text{if } \Gamma(x) = \forall \alpha_1, \ldots, \alpha_n \,.\, \tau$$

- Where φ replaces all occurrences of $\alpha_1, \ldots, \alpha_n$ by monotypes $\tau_1, \ldots, \tau_n$
- Note: Monomorphic rule special case:

$$\frac{}{\Gamma \mid\!- x : \tau} \quad \text{if } \Gamma(x) = \tau$$

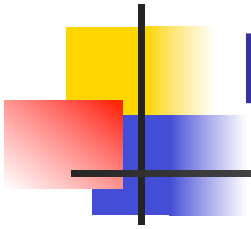
- Constants treated same way

Fun Rule Stays the Same

- fun rule:

$$\frac{\{x : \tau_1\} + \Gamma \text{ |- } e : \tau_2}{\Gamma \text{ |- fun } x \text{ -> } e : \tau_1 \rightarrow \tau_2}$$

- Types τ_1, τ_2 monomorphic
- Function argument must always be used at same type in function body



Polymorphic Example

- Assume additional constants:
- hd :$\forall\alpha.\ \alpha$ list -> α
- tl: $\forall\alpha.\ \alpha$ list -> α list
- is_empty : $\forall\alpha.\ \alpha$ list -> bool
- :: : $\forall\alpha.\ \alpha$ -> α list -> α list
- [] : $\forall\alpha.\ \alpha$ list

Polymorphic Example: Let Rec Rule

?

```
{} |- let rec length =
        fun l -> if is_empty l then 0
                 else 1 + length (tl l)
  in length (2 :: []) + length(true :: []) : int
```

Polymorphic Example: Let Rec Rule

- Show: (1) (2)

{length:α list -> int} {length:$\forall\alpha.\ \alpha$ list -> int}

|- fun l -> … |- length (2 :: []) +

: α list -> int length(true :: []) : int

{} |- let rec length =

fun l -> if is_empty l then 0

else 1 + length (tl l)

in length (2 :: []) + length(true :: []) : int

Polymorphic Example (1)

- Show:

$$\frac{\quad ? \quad}{\begin{array}{l}\{\text{length}:\alpha\ \text{list} \rightarrow \text{int}\} \vdash \\ \text{fun l -> if is\_empty l then 0} \\ \qquad\qquad\qquad \text{else 1 + length (tl l)} \\ : \alpha\ \text{list} \rightarrow \text{int}\end{array}}$$

Polymorphic Example (1): Fun Rule

- Show: (3)

{length:α list -> int, l: α list } |-

if is_empty l then 0

else length (hd l) + length (tl l) : int

{length:α list -> int} |-

fun l -> if is_empty l then 0

else 1 + length (tl l)

: α list -> int

Polymorphic Example (3)

- Let Γ ={length:α list -> int, l: α list }
- Show

$$\frac{?}{\Gamma \vdash \text{if is\_empty l then 0 else 1 + length (tl l)} : \text{int}}$$

Polymorphic Example (3):IfThenElse

- Let Γ ={length:α list -> int, l: α list }
- Show

$$\frac{\begin{array}{lll} (4) & (5) & (6) \\ \Gamma\text{|- is\_empty l} & \Gamma\text{|- 0:int} & \Gamma\text{|- 1 +} \\ \text{: bool} & & \text{length (tl l) : int} \end{array}}{\begin{array}{l} \Gamma\text{|- if is\_empty l then 0} \\ \quad \text{else 1 + length (tl l) : int} \end{array}}$$

Polymorphic Example (4)

- Let Γ ={length:α list -> int, l: α list }
- Show

$$\frac{?}{\Gamma \vdash \text{is\_empty l} : \text{bool}}$$

Polymorphic Example (4):Application

- Let Γ ={length:α list -> int, l: α list }
- Show

$$\frac{\dfrac{?}{\Gamma \vdash \text{is\_empty} : \alpha \text{ list} \rightarrow \text{bool}} \qquad \dfrac{?}{\Gamma \vdash \text{l} : \alpha \text{ list}}}{\Gamma \vdash \text{is\_empty l} : \text{bool}}$$

Polymorphic Example (4)

- Let Γ ={length:α list -> int, l: α list }
- Show

By Const since α list -> bool is instance of $\forall\alpha$. α list -> bool ?

$$\frac{\Gamma \vdash \text{is\_empty} : \alpha \text{ list} \to \text{bool} \qquad \Gamma \vdash \text{l} : \alpha \text{ list}}{\Gamma \vdash \text{is\_empty l} : \text{bool}}$$

Polymorphic Example (4)

- Let Γ ={length:α list -> int, l: α list }
- Show

By Const since α list -> bool is instance of $\forall\alpha$. α list -> bool

By Variable $\Gamma(l) = \alpha$ list

$$\frac{\Gamma \vdash \text{is\_empty} : \alpha \text{ list} \to \text{bool} \qquad \Gamma \vdash l : \alpha \text{ list}}{\Gamma \vdash \text{is\_empty } l : \text{bool}}$$

- This finishes (4)

Polymorphic Example (5):Const

- Let Γ ={length:α list -> int, l: α list }
- Show

By Const Rule

$$\frac{}{\Gamma \vdash 0:\text{int}}$$

Polymorphic Example (6):Arith Op

- Let Γ ={length:α list -> int, l: α list }
- Show

$$\dfrac{\dfrac{\dfrac{}{\Gamma \vdash 1 : \text{int}}\ \text{By Const} \qquad \dfrac{\dfrac{}{\Gamma \vdash \text{length} : \alpha\ \text{list} \rightarrow \text{int}}\ \text{By Variable} \qquad \dfrac{}{\Gamma \vdash (\text{tl}\ l) : \alpha\ \text{list}}\ (7)}{\Gamma \vdash \text{length}\ (\text{tl}\ l) : \text{int}}}{\Gamma \vdash 1 + \text{length}\ (\text{tl}\ l) : \text{int}}$$

Polymorphic Example (7):App Rule

- Let Γ ={length:α list -> int, l: α list }
- Show

$$\frac{\dfrac{}{\Gamma \vdash \text{tl} : \alpha\ \text{list} \rightarrow \alpha\ \text{list}}\ \text{By Const} \qquad \dfrac{}{\Gamma \vdash \text{l} : \alpha\ \text{list}}\ \text{By Variable}}{\Gamma \vdash (\text{tl l}) : \alpha\ \text{list}}$$

By Const since α list -> α list is instance of

$\forall\alpha.\ \alpha$ list -> α list

Polymorphic Example: (2) by ArithOp

- Let Γ' = {length:$\forall \alpha.\ \alpha$ list -> int}
- Show:

(8) (9)

Γ' |- Γ' |-

length (2 :: []) :int length(true :: []) : int

{length: $\alpha.\ \alpha$ list -> int}

|- length ((::) 2 []) + length((::) true []) : int

Polymorphic Example: (8)AppRule

- Let Γ' = {length:$\forall \alpha.\ \alpha$ list -> int}
- Show:

$$\frac{\Gamma' \vdash \text{length} : \text{int list} \rightarrow \text{int} \quad \Gamma' \vdash (2 :: []) : \text{int list}}{\Gamma' \vdash \text{length}\ (2 :: []) : \text{int}}$$

Polymorphic Example: (8)AppRule

- Let Γ' = {length:$\forall \alpha.\ \alpha$ list -> int}
- Show:

By Var since int list -> int is instance of

$\forall\alpha.\ \alpha$ list -> int

$$\frac{\Gamma' \text{ |- length : int list ->int} \qquad \Gamma' \text{ |- ((::)2 []):int}\ \text{list}}{\Gamma' \text{ |- length ((::) 2 []) :int}} \quad (10)$$

Polymorphic Example: (10)AppRule

- Let Γ' = {length:$\forall \alpha.\ \alpha$ list -> int}
- Show:
- By Const since α list is instance of

  $\forall\alpha.\ \alpha$ list

$$\frac{\Gamma' \text{ |-((::) 2) : int list -> int list} \quad \overline{\Gamma' \text{ |- [] : int list}}\ (11)}{\Gamma' \text{ |- ((::) 2 []) :int list}}$$

Polymorphic Example: (11)AppRule

- Let Γ' = {length:$\forall \alpha.\ \alpha$ list -> int}
- Show:
- By Const since α list is instance of $\forall\alpha.\ \alpha$ list

$$\frac{\dfrac{\Gamma' \vdash (::) : \text{int} \to \text{int list} \to \text{int list} \qquad \Gamma' \vdash 2 : \text{int}}{\Gamma' \vdash ((::)\ 2) : \text{int list} \to \text{int list}}\ \text{By Const}}{}$$

Polymorphic Example: (9)AppRule

- Let Γ' = {length:$\forall\ \alpha.\ \alpha$ list -> int}
- Show:

$$\frac{\Gamma' \text{ |- length:bool list ->int} \qquad \Gamma' \text{ |- ((::) true []):bool list}}{\Gamma' \text{ |- length ((::) true []) :int}}$$

Polymorphic Example: (9)AppRule

- Let Γ' = {length:$\forall \alpha.\ \alpha$ list -> int}
- Show:

By Var since bool list -> int is instance of

$\forall\alpha.\ \alpha$ list -> int

(12)

$$\frac{\Gamma' \vdash \text{length}:\text{bool list} \rightarrow \text{int} \qquad \Gamma' \vdash ((::)\ \text{true}\ []):\text{bool list}}{\Gamma' \vdash \text{length}\ ((::)\ \text{true}\ []) : \text{int}}$$

Polymorphic Example: (12)AppRule

- Let Γ' = {length:$\forall \alpha.\ \alpha$ list -> int}
- Show:
- By Const since α list is instance of

$\forall\alpha.\ \alpha$ list

(13)

$$\frac{\Gamma' \text{ |-((::)true):bool list ->bool list} \quad \overline{\Gamma' \text{ |- []:bool list}}}{\Gamma' \text{ |- ((::) true []) :bool list}}$$

Polymorphic Example: (13)AppRule

- Let Γ' = {length:$\forall \alpha.\ \alpha$ list -> int}
- Show:

$$\frac{\dfrac{\Gamma' \vdash (::):\text{bool} \rightarrow \text{bool list} \rightarrow \text{bool list} \quad \text{By Const since bool list is instance of } \forall\alpha.\ \alpha \text{ list}}{\Gamma' \vdash \text{true} : \text{bool}} \quad \text{By Const}}{\Gamma' \vdash ((::)\ \text{true}) : \text{bool list} \rightarrow \text{bool list}}$$