Programming Languages and Compilers (CS 421)

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Based in part on slides by Mattox Beckman, as updated by Vikram Adve and Gul Agha

Mapping Functions Over Lists

```
# let rec map f list =
  match list
  with [] -> []
  | (h::t) -> (f h) :: (map f t);;
val map : ('a -> 'b) -> 'a list -> 'b list = <fun>
# map plus_two fib5;;
- : int list = [10; 7; 5; 4; 3; 3]
# map (fun x -> x - 1) fib6;;
: int list = [12; 7; 4; 2; 1; 0; 0]
```

Mapping Recursion

- One common form of structural recursion applies a function to each element in the structure

```
# let rec doubleList list = match list
  with [ ] -> [ ]
  | x::xs -> 2 * x :: doubleList xs;;
val doubleList : int list -> int list = <fun>
# doubleList [2;3;4];;
- : int list = [4; 6; 8]
```

Mapping Recursion

- Can use the higher-order recursive map function instead of direct recursion

```
# let doubleList list =
   List.map (fun x -> 2 * x) list;;
val doubleList : int list -> int list = <fun>
# doubleList [2;3;4];;
- : int list = [4; 6; 8]
```

- Same function, but no explicit rec

Your turn now

Write a function

make_app : ((‘a -> ‘b) * ‘a) list -> ‘b list

that takes a list of function – input pairs and gives the result of applying each function to its argument. Use map, no explicit recursion.

let make_app l =

Folding Recursion

- Another common form “folds” an operation over the elements of the structure

```
# let rec multList list = match list
  with [ ] -> 1
  | x::xs -> x * multList xs;;
val multList : int list -> int = <fun>
# multList [2;4;6];;
- : int = 48
```

- Computes (2 * (4 * (6 * 1)))

Folding Functions over Lists

How are the following functions similar?

```
# let rec sumList list = match list with
  [ ] -> 0 | x::xs -> x + sumList xs;;
val sumList : int list -> int = <fun>
# sumList [2;3;4];;
- : int = 9
# let rec multList list = match list with
  [ ] -> 1 | x::xs -> x * multList xs;;
val multList : int list -> int = <fun>
# multList [2;3;4];;
- : int = 24
```

Folding Functions over Lists

How are the following functions similar?

```
# let rec sumList list = match list with
  [ ] -> 0 | x::xs -> x + sumList xs;;
val sumList : int list -> int = <fun>
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- : int = 9
# let rec multList list = match list with
  [ ] -> 1 | x::xs -> x * multList xs;;
val multList : int list -> int = <fun>
# multList [2;3;4];;
- : int = 24
```

Base Case

Folding Functions over Lists

How are the following functions similar?

```
# let rec sumList list = match list with
  [ ] -> 0 | x::xs -> x + sumList xs;;
val sumList : int list -> int = <fun>
# sumList [2;3;4];;
- : int = 9
# let rec multList list = match list with
  [ ] -> 1 | x::xs -> x * multList xs;;
val multList : int list -> int = <fun>
# multList [2;3;4];;
- : int = 24
```

Recursive Call

Folding Functions over Lists

How are the following functions similar?

```
# let rec sumList list = match list with
  [ ] -> 0 | x::xs -> x + sumList xs;;
val sumList : int list -> int = <fun>
# sumList [2;3;4];;
- : int = 9
# let rec multList list = match list with
  [ ] -> 1 | x::xs -> x * multList xs;;
val multList : int list -> int = <fun>
# multList [2;3;4];;
- : int = 24
```

Head Element

Folding Functions over Lists

How are the following functions similar?

```
# let rec sumList list = match list with
  [ ] -> 0 | x::xs -> x + sumList xs;;
val sumList : int list -> int = <fun>
# sumList [2;3;4];;
- : int = 9
# let rec multList list = match list with
  [ ] -> 1 | x::xs -> x * multList xs;;
val multList : int list -> int = <fun>
# multList [2;3;4];;
- : int = 24
```

Combining Operation

Folding Functions over Lists

How are the following functions similar?

```
# let rec sumList list = match list with
  [ ] -> 0 | x::xs -> x + Rec value ;
val sumList : int list -> int = <fun>
# sumList [2;3;4];;
- : int = 9
# let rec multList list = match list with
  [ ] -> 1 | x::xs -> x * Rec value ;
val multList : int list -> int = <fun>
# multList [2;3;4];;
- : int = 24R
```

Combining Operation

Recursing over lists

The Primitive Recursion Fairy

```
# let rec fold_right f list b =
  match list
  with [] -> b
  | (x :: xs) -> f x (fold_right f xs b);;
val fold_right : ('a -> 'b -> 'b) -> 'a list -> 'b -> 'b =
  <fun>
# fold_right
    (fun s -> fun () -> print_string s)
    ["hi"; "there"]
    ();;
therehi- : unit = ()
```

Folding Recursion

- multList folds to the right
- Same as:

```
# let multList list =
    List.fold_right
    (fun x -> fun rv -> x * rv)
    list 1;;
val multList : int list -> int = <fun>
# multList [2;4;6];;
- : int = 48
```

Encoding Recursion with Fold

```
# let rec append list1 list2 = match list1 with
  [ ] -> list2 | x::xs -> x :: append xs list2;;
val append : 'a list -> 'a list -> 'a list = <fun>
```

Base Case   Operation   Recursive Call

```
# let append list1 list2 =
   fold_right (fun x rv -> x :: rv) list1 list2;;
val append : 'a list -> 'a list -> 'a list = <fun>
# append [1;2;3] [4;5;6];;
 - : int list = [1; 2; 3; 4; 5; 6]
```

Your turn now

Try Problem 1 on ML2

Question

```
let rec length l =
    match l with [] -> 0
     | (a :: bs) -> 1 + length bs
```

- How do you write length with fold_right, but no explicit recursion?

Question

```
let rec length l =
    match l with [] -> 0
     | (a :: bs) -> 1 + length bs
```

- How do you write length with fold_right, but no explicit recursion?

```
let length list =
 List.fold_right (fun x -> fun n ->  n + 1) list 0
```

Map from Fold

```
# let map f list =
 fold_right (fun x -> fun y -> f x :: y) list
   [ ];;
val map : ('a -> 'b) -> 'a list -> 'b list =
   <fun>
# map ((+)1) [1;2;3];;
- : int list = [2; 3; 4]
```

- Can you write fold_right (or fold_left) with just map? How, or why not?

Iterating over lists

```
# let rec fold_left f a list =
  match list
  with [] -> a
  | (x :: xs) -> fold_left f (f a x) xs;;
val fold_left : ('a -> 'b -> 'a) -> 'a -> 'b list -> 'a =
   <fun>
# fold_left
   (fun () -> print_string)
   ()
   ["hi"; "there"];;
hithere- : unit = ()
```

Encoding Tail Recursion with fold_left

```
# let prod list = let rec prod_aux l acc =
       match l with [] -> acc
       | (y :: rest) -> prod_aux rest (acc * y)
      in prod_aux list 1;;
val prod : int list -> int = <fun>
```

Init Acc Value | Recursive Call | Operation

```
# let prod list =
  List.fold_left (fun acc y -> acc * y) 1 list;;
val prod: int list -> int = <fun>
# prod [4;5;6];;
 - : int =120
```

Question

```
let length l =
    let rec length_aux list n =
    match list with [] -> n
     | (a :: bs) -> length_aux bs (n + 1)
in length_aux l 0
```

- How do you write length with fold_left, but no explicit recursion?

Question

```
let length l =
    let rec length_aux list n =
    match list with [] -> n
     | (a :: bs) -> length_aux bs (n + 1)
in length_aux l 0
```

- How do you write length with fold_left, but no explicit recursion?

```
let length list =
 List.fold_left (fun n -> fun x -> n + 1) 0 list
```

Folding

```
# let rec fold_left f a list = match list
  with [] -> a | (x :: xs) -> fold_left f (f a x) xs;;
val fold_left : ('a -> 'b -> 'a) -> 'a -> 'b list -> 'a =
   <fun>
```

fold_left f a [x_1; x_2;…;x_n] = f(…(f (f a x_1) x_2)…)x_n

```
# let rec fold_right f list b = match list
  with [ ] -> b | (x :: xs) -> f x (fold_right f xs b);;
val fold_right : ('a -> 'b -> 'b) -> 'a list -> 'b -> 'b =
   <fun>
```

fold_right f [x_1; x_2;…;x_n] b = f x_1(f x_2 (…(f x_n b)…))

Recall

```
# let rec poor_rev list = match list
  with [] -> []
     | (x::xs) -> poor_rev xs @ [x];;
val poor_rev : 'a list -> 'a list = <fun>
```

- What is its running time?

Quadratic Time

- Each step of the recursion takes time proportional to input
- Each step of the recursion makes only one recursive call.
- List example:

```
# let rec poor_rev list = match list
  with [] -> []
     | (x::xs) -> poor_rev xs @ [x];;
val poor_rev : 'a list -> 'a list = <fun>
```

Tail Recursion - Example

```
# let rec rev_aux list revlist =
  match list with [ ] -> revlist
  | x :: xs -> rev_aux xs (x::revlist);;
val rev_aux : 'a list -> 'a list -> 'a list = <fun>

# let rev list = rev_aux list [ ];;
val rev : 'a list -> 'a list = <fun>
```

- What is its running time?

Comparison

- poor_rev [1,2,3] =
- (poor_rev [2,3]) @ [1] =
- ((poor_rev [3]) @ [2]) @ [1] =
- (((poor_rev [ ]) @ [3]) @ [2]) @ [1] =
- (([ ] @ [3]) @ [2]) @ [1]) =
- ([3] @ [2]) @ [1] =
- (3:: ([ ] @ [2])) @ [1] =
- [3,2] @ [1] =
- 3 :: ([2] @ [1]) =
- 3 :: (2:: ([ ] @ [1])) = [3, 2, 1]

Comparison

- rev [1,2,3] =
- rev_aux [1,2,3] [ ] =
- rev_aux [2,3] [1] =
- rev_aux [3] [2,1] =
- rev_aux [ ] [3,2,1] = [3,2,1]

Folding - Tail Recursion

```
- # let rev list =
-       fold_left
-         (fun l -> fun x -> x :: l)     //comb op
          []              //accumulator cell
          list
```

Folding

- Can replace recursion by fold_right in any forward primitive recursive definition
  - Primitive recursive means it only recurses on immediate subcomponents of recursive data structure
- Can replace recursion by fold_left in any tail primitive recursive definition

Continuations

- A programming technique for all forms of “non-local” control flow:
  - non-local jumps
  - exceptions
  - general conversion of non-tail calls to tail calls
- Essentially it’s a higher-order function version of GOTO

Continuations

- Idea: Use functions to represent the control flow of a program
- Method: Each procedure takes a function as an extra argument to which to pass its result; outer procedure “returns” no result
- Function receiving the result called a continuation
- Continuation acts as “accumulator” for work still to be done

Continuation Passing Style

- Writing procedures such that all procedure calls take a continuation to which to give (pass) the result, and return no result, is called continuation passing style (CPS)

Continuation Passing Style

- A compilation technique to implement non-local control flow, especially useful in interpreters.
- A formalization of non-local control flow in denotational semantics
- Possible intermediate state in compiling functional code

Why CPS?

- Makes order of evaluation explicitly clear
- Allocates variables (to become registers) for each step of computation
- Essentially converts functional programs into imperative ones
  - Major step for compiling to assembly or byte code
- Tail recursion easily identified
- Strict forward recursion converted to tail recursion
  - At the expense of building large closures in heap

Example

- Simple reporting continuation:

```
# let report x = (print_int x; print_newline( ) );;
val report : int -> unit = <fun>
```

- Simple function using a continuation:

```
# let addk a b k = k (a + b);;
val addk : int -> int -> (int -> ’a) -> ’a = <fun>
# addk 22 20 report;;
2
- : unit = ()
```

Simple Functions Taking Continuations

- Given a primitive operation, can convert it to pass its result forward to a continuation
- Examples:

```
# let subk x y k = k(x + y);;
val subk : int -> int -> (int -> 'a) -> 'a = <fun>
# let eqk x y k = k(x = y);;
val eqk : 'a -> 'a -> (bool -> 'b) -> 'b = <fun>
# let timesk x y k = k(x * y);;
val timesk : int -> int -> (int -> 'a) -> 'a = <fun>
```

Your turn now

Try Problem 5 on MP4

Try modk

Nesting Continuations

```
# let add_three x y z = (x + y) + z;;
val add_three : int -> int -> int -> int = <fun>
# let add_three x y z= let p = x + y in  p + z;;
val add_three : int -> int -> int -> int = <fun>
# let add_three_k x y z k =
   addk x y (fun p -> addk p z k );;
val add_three_k : int -> int -> int -> (int -> 'a)
  -> 'a = <fun>
```

add_three: a different order

- # let add_three x y z = x + (y + z);;
- How do we write add_three_k to use a different order?
- let add_three_k x y z k =

Your turn now

Try Problem 6 on MP4

Recursive Functions

- Recall:

```
# let rec factorial n =
    if n = 0 then 1 else n * factorial (n - 1);;
  val factorial : int -> int = <fun>
# factorial 5;;
- : int = 120
```

Recursive Functions

```
# let rec factorial n =
   let b = (n = 0) in (* First computation *)
   if b then 1 (* Returned value *)
   else let s = n – 1 in (* Second computation *)
          let r = factorial s in  (* Third computation *)
          n * r in (* Returned value *) ;;
val factorial : int -> int = <fun>
# factorial 5;;
- : int = 120
```

Recursive Functions

```
# let rec factorialk n k =
   eqk n 0
   (fun b ->  (* First computation *)
    if b then k 1 (* Passed value *)
    else subk n 1  (* Second computation *)
    (fun s -> factorialk s  (* Third computation *)
    (fun r -> timesk n r k))) (* Passed value *)
val factorialk : int -> int = <fun>
# factorialk 5 report;;
120
- : unit = ()
```

Recursive Functions

- To make recursive call, must build intermediate continuation to
  - take recursive value: r
  - build it to final result: n * r
  - And pass it to final continuation:
  -   times n r k = k (n * r)

Example: CPS for length

```
let rec length list = match list with [] -> 0
    | (a :: bs) -> 1 + length bs
```

What is the let-expanded version of this?

Example: CPS for length

```
let rec length list = match list with [] -> 0
    | (a :: bs) -> 1 + length bs
```

What is the let-expanded version of this?

```
let rec length list = match list with [] -> 0
    | (a :: bs) -> let r1 = length bs in 1 + r1
```

Example: CPS for length

```
#let rec length list = match list with [] -> 0
   | (a :: bs) -> let r1 = length bs in 1 + r1
```

What is the CSP version of this?

Example: CPS for length

```
#let rec length list = match list with [] -> 0
   | (a :: bs) -> let r1 = length bs in 1 + r1
```

What is the CSP version of this?

```
#let rec lengthk list k = match list with [ ] -> k 0
   | x :: xs -> lengthk xs (fun r -> addk r 1 k);;
val lengthk : 'a list -> (int -> 'b) -> 'b = <fun>
# lengthk [2;4;6;8] report;;
4
- : unit = ()
```

Your turn now

Try Problem 8 on MP4

CPS for Higher Order Functions

- In CPS, every procedure / function takes a continuation to receive its result
- Procedures passed as arguments take continuations
- Procedures returned as results take continuations
- CPS version of higher-order functions must expect input procedures to take continuations

Example: all

```
#let rec all p l = match l with [] -> true
   | (x :: xs) -> let b = p x in
      if b then all p xs else false
val all : ('a -> bool) -> 'a list -> bool = <fun>
```

- What is the CPS version of this?

Example: all

```
#let rec all p l = match l with [] -> true
   | (x :: xs) -> let b = p x in
      if b then all p xs else false
val all : ('a -> bool) -> 'a list -> bool = <fun>
```

- What is the CPS version of this?

```
#let rec allk pk l k =
```

Example: all

```
#let rec all p l = match l with [] -> true
    | (x :: xs) -> let b = p x in
       if b then all p xs else false
val all : ('a -> bool) -> 'a list -> bool = <fun>
```

- What is the CPS version of this?

```
#let rec allk pk l k = match l with [] ->     true
```

Example: all

```
#let rec all p l = match l with [] -> true
    | (x :: xs) -> let b = p x in
       if b then all p xs else false
val all : ('a -> bool) -> 'a list -> bool = <fun>
```

- What is the CPS version of this?

```
#let rec allk pk l k = match l with [] ->  k true
```

Example: all

```
#let rec all p l = match l with [] -> true
    | (x :: xs) -> let b = p x in
       if b then all p xs else false
val all : ('a -> bool) -> 'a list -> bool = <fun>
```

- What is the CPS version of this?

```
#let rec allk pk l k = match l with [] ->  k true
 | (x :: xs) ->
```

Example: all

```
#let rec all p l = match l with [] -> true
    | (x :: xs) -> let b = p x in
       if b then all p xs else false
val all : ('a -> bool) -> 'a list -> bool = <fun>
```

- What is the CPS version of this?

```
#let rec allk pk l k = match l with [] ->  k true
 | (x :: xs) -> pk x
```

Example: all

```
#let rec all p l = match l with [] -> true
    | (x :: xs) -> let b = p x in
       if b then all p xs else false
val all : ('a -> bool) -> 'a list -> bool = <fun>
```

- What is the CPS version of this?

```
#let rec allk pk l k = match l with [] ->  k true
 | (x :: xs) -> pk x
         (fun b -> if b then                  else
     )
```

Example: all

```
#let rec all p l = match l with [] -> true
    | (x :: xs) -> let b = p x in
       if b then all p xs else false
val all : ('a -> bool) -> 'a list -> bool = <fun>
```

- What is the CPS version of this?

```
#let rec allk pk l k = match l with [] ->  k true
 | (x :: xs) -> pk x
         (fun b -> if b then allk pk xs k else k
false)
val allk : ('a -> (bool -> 'b) -> 'b) -> 'a list ->
(bool -> 'b) -> 'b = <fun>
```

Terms

- A function is in Direct Style when it returns its result back to the caller.
- A Tail Call occurs when a function returns the result of another function call without any more computations (eg tail recursion)
- A function is in Continuation Passing Style when it, and every function call in it, passes its result to another function.
- Instead of returning the result to the caller, we pass it forward to another function.

Terminology

- Tail Position: A subexpression s of expressions e, such that if evaluated, will be taken as the value of e
  - if (x>3) then $\boxed{\text{x + 2}}$ else $\boxed{\text{x - 4}}$
  - let x = 5 in $\boxed{\text{x + 4}}$
- Tail Call: A function call that occurs in tail position
  - if (h x) then $\boxed{\text{f x}}$ else $\boxed{\text{(x } \underline{+} \text{ g x)}}$

Terminology

- Available: A function call that can be executed by the current expression
- The fastest way to be unavailable is to be guarded by an abstraction (anonymous function, lambda lifted).
  - if $\boxed{\text{(h x)}}$ then $\boxed{\text{f x}}$ else $\boxed{\text{(x + g x)}}$
  - if $\boxed{\text{(h x)}}$ then (fun x -> f x) else $\boxed{\text{(g (x + x))}}$

Not available

CPS Transformation

- Step 1: Add continuation argument to any function definition:
  - let f arg = e ⇒ let f arg k = e
  - Idea: Every function takes an extra parameter saying where the result goes
- Step 2: A simple expression in tail position should be passed to a continuation instead of returned:
  - return a ⇒ k a
  - Assuming a is a constant or variable.
  - "Simple" = "No available function calls."

CPS Transformation

- Step 3: Pass the current continuation to every function call in tail position
  - return f arg ⇒ f arg k
  - The function "isn't going to return," so we need to tell it where to put the result.

CPS Transformation

- Step 4: Each function call not in tail position needs to be converted to take a new continuation (containing the old continuation as appropriate)
  - return op (f arg) ⇒ f arg (fun r -> k(op r))
  - op represents a primitive operation
  - return f(g arg) ⇒ g arg (fun r-> f r k)

Example

Before:

```
let rec add_list lst =
match lst with
  [ ] -> 0
| 0 :: xs -> add_list xs
| x :: xs -> (+) x
    (add_list xs);;
```

After:

```
let rec add_listk lst k =
                    (* rule 1 *)
match lst with
| [ ] -> k 0 (* rule 2 *)
| 0 :: xs -> add_listk xs k
                    (* rule 3 *)
| x :: xs -> add_listk xs
         (fun r -> k ((+) x r));;
                 (* rule 4 *)
```

CPS for sum

```
# let rec sum list = match list with [ ] -> 0
   | x :: xs -> x + sum xs ;;
val sum : int list -> int = <fun>
```

CPS for sum

```
# let rec sum list = match list with [ ] -> 0
   | x :: xs -> x + sum xs ;;
val sum : int list -> int = <fun>
# let rec sum list = match list with [ ] -> 0
   | x :: xs -> let r1 = sum xs  in x + r1;;
```

CPS for sum

```
# let rec sum list = match list with [ ] -> 0
   | x :: xs -> x + sum xs ;;
val sum : int list -> int = <fun>
# let rec sum list = match list with [ ] -> 0
   | x :: xs -> let r1 = sum xs  in x + r1;;
val sum : int list -> int = <fun>
# let rec sumk list k = match list with [ ] -> k 0
    | x :: xs -> sumk xs  (fun r1 -> addk x r1 k);;
```

CPS for sum

```
# let rec sum list = match list with [ ] -> 0
   | x :: xs -> x + sum xs ;;
val sum : int list -> int = <fun>
# let rec sum list = match list with [ ] -> 0
   | x :: xs -> let r1 = sum xs  in x + r1;;
val sum : int list -> int = <fun>
# let rec sumk list k = match list with [ ] -> k 0
    | x :: xs -> sumk xs  (fun r1 -> addk x r1 k);;
val sumk : int list -> (int -> 'a) -> 'a = <fun>
# sumk [2;4;6;8] report;;
20
- : unit = ()
```

Other Uses for Continuations

- CPS designed to preserve order of evaluation
- Continuations used to express order of evaluation
- Can be used to change order of evaluation
- Implements:
  - Exceptions and exception handling
  - Co-routines
  - (pseudo, aka green) threads

Exceptions - Example

```
# exception Zero;;
exception Zero
# let rec list_mult_aux list =
    match list with [ ] -> 1
    | x :: xs ->
     if x = 0 then raise Zero
              else x * list_mult_aux xs;;
val list_mult_aux : int list -> int = <fun>
```

Exceptions - Example

```
# let list_mult list =
    try list_mult_aux list with Zero -> 0;;
val list_mult : int list -> int = <fun>
# list_mult [3;4;2];;
- : int = 24
# list_mult [7;4;0];;
- : int = 0
# list_mult_aux [7;4;0];;
Exception: Zero.
```

Exceptions

- When an exception is raised
  - The current computation is aborted
  - Control is "thrown" back up the call stack until a matching handler is found
  - All the intermediate calls waiting for a return values are thrown away

Implementing Exceptions

```
# let multkp m n k =
   let r = m * n in
    (print_string "product result: ";
     print_int r; print_string "\n";
     k r);;
val multkp : int -> int -> (int -> 'a) -> 'a
  = <fun>
```

Implementing Exceptions

```
# let rec list_multk_aux list k kexcp =
    match list with [ ] -> k 1
    | x :: xs -> if x = 0 then  kexcp  0
      else list_multk_aux xs
           (fun r -> multkp x r k) kexcp;;
val list_multk_aux : int list -> (int -> 'a) -> (int -> 'a)
   -> 'a = <fun>
# let rec list_multk list k = list_multk_aux list  k  k;;
val list_multk : int list -> (int -> 'a) -> 'a = <fun>
```

Implementing Exceptions

```
# list_multk [3;4;2] report;;
product result: 2
product result: 8
product result: 24
24
- : unit = ()
# list_multk [7;4;0] report;;
0
- : unit = ()
```