

Programming Languages and Compilers (CS 421)

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Based in part on slides by Mattox Beckman, as updated by Vikram Adve and Gul Agha

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Lambda Calculus - Motivation

- Aim is to capture the essence of functions, function applications, and evaluation
- λ -calculus is a theory of computation
- "The Lambda Calculus: Its Syntax and Semantics". H. P. Barendregt. North Holland, 1984

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Lambda Calculus - Motivation

- All *sequential programs* may be viewed as functions from input (initial state and input values) to output (resulting state and output values).
- λ -calculus is a mathematical formalism of functions and functional computations
- Two flavors: typed and untyped

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Untyped λ -Calculus

- Only three kinds of expressions:
 - Variables: x, y, z, w, \dots
 - Abstraction: $\lambda x. e$
(Function creation, think `fun x -> e`)
 - Application: $e_1 e_2$

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Untyped λ -Calculus Grammar

- Formal BNF Grammar:
 - $\langle \text{expression} \rangle ::= \langle \text{variable} \rangle$
 | $\langle \text{abstraction} \rangle$
 | $\langle \text{application} \rangle$
 | $(\langle \text{expression} \rangle)$
 - $\langle \text{abstraction} \rangle ::= \lambda \langle \text{variable} \rangle . \langle \text{expression} \rangle$
 - $\langle \text{application} \rangle ::= \langle \text{expression} \rangle \langle \text{expression} \rangle$

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Untyped λ -Calculus Terminology

- **Occurrence**: a location of a subterm in a term
- **Variable binding**: $\lambda x. e$ is a binding of x in e
- **Bound occurrence**: all occurrences of x in $\lambda x. e$
- **Free occurrence**: one that is not bound
- **Scope of binding**: in $\lambda x. e$, all occurrences in e not in a subterm of the form $\lambda x. e'$ (same x)
- **Free variables**: all variables having free occurrences in a term

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Example

- Label occurrences and scope:

$(\lambda x. \lambda y. y (\lambda x. x y) x) x$

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Example

- Label occurrences and scope:

$(\lambda x. y \lambda y. y (\lambda x. x y) x) x$

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Untyped λ -Calculus

- How do you compute with the λ -calculus?
- Roughly speaking, by substitution:
 - $(\lambda x. e_1) e_2 \Rightarrow^* e_1 [e_2 / x]$
- * Modulo all kinds of subtleties to avoid free variable capture

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Transition Semantics for λ -Calculus

- $$\frac{E \rightarrow E''}{E E' \rightarrow E'' E'}$$
- Application (version 1 - Lazy Evaluation)

$$(\lambda x. E) E' \rightarrow E[E'/x]$$
 - Application (version 2 - Eager Evaluation)

$$\frac{E' \rightarrow E''}{(\lambda x. E) E' \rightarrow (\lambda x. E) E''}$$
- $$\overline{(\lambda x. E) V \rightarrow E[V/x]}$$
- V - variable or abstraction (value)

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How Powerful is the Untyped λ -Calculus?

- The untyped λ -calculus is Turing Complete
 - Can express any sequential computation
- Problems:
 - How to express basic data: booleans, integers, etc?
 - How to express recursion?
 - Constants, if_then_else, etc, are conveniences; can be added as syntactic sugar

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Typed vs Untyped λ -Calculus

- The *pure* λ -calculus has no notion of type: $(f f)$ is a legal expression
- Types restrict which applications are valid
- Types are not syntactic sugar! They disallow some terms
- Simply typed λ -calculus is less powerful than the untyped λ -Calculus: NOT Turing Complete (no recursion)

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Uses of λ -Calculus

- Typed and untyped λ -calculus used for theoretical study of sequential programming languages
- Sequential programming languages are essentially the λ -calculus, extended with predefined constructs, constants, types, and syntactic sugar
- Ocaml is close to the λ -Calculus:

$$\text{fun } x \rightarrow \text{exp} \rightarrow \lambda x. \text{exp}$$

$$\text{let } x = e_1 \text{ in } e_2 \rightarrow (\lambda x. e_2)e_1$$

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α Conversion

- α -conversion:

$$\lambda x. \text{exp} \rightarrow \lambda y. (\text{exp } [y/x])$$
- Provided that
 - y is not free in exp
 - No free occurrence of x in exp becomes bound in exp when replaced by y

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α Conversion Non-Examples

- Error: y is not free in termsecond

$$\lambda x. x y \rightarrow \lambda y. y y$$
- Error: free occurrence of x becomes bound in wrong way when replaced by y

$$\lambda x. \underbrace{\lambda y. x y}_{\text{exp}} \rightarrow \lambda y. \underbrace{\lambda y. y y}_{\text{exp}[y/x]}$$

But $\lambda x. (\lambda y. y) x \rightarrow \lambda y. (\lambda y. y) y$
 And $\lambda y. (\lambda y. y) y \rightarrow \lambda x. (\lambda y. y) x$

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Congruence

- Let \sim be a relation on lambda terms. \sim is a **congruence** if
 - it is an equivalence relation
 - If $e_1 \sim e_2$ then
 - $(e e_1) \sim (e e_2)$ and $(e_1 e) \sim (e_2 e)$
 - $\lambda x. e_1 \sim \lambda x. e_2$

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α Equivalence

- α equivalence is the smallest congruence containing α conversion
- One usually treats α -equivalent terms as equal - i.e. use α equivalence classes of terms

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Example

- Show: $\lambda x. (\lambda y. y x) x \sim \lambda y. (\lambda x. x y) y$
- $\lambda x. (\lambda y. y x) x \rightarrow \lambda z. (\lambda y. y z) z$ so
 $\lambda x. (\lambda y. y x) x \sim \lambda z. (\lambda y. y z) z$
 - $(\lambda y. y z) \rightarrow (\lambda x. x z)$ so
 $(\lambda y. y z) \sim (\lambda x. x z)$ so
 $\lambda z. (\lambda y. y z) z \sim \lambda z. (\lambda x. x z) z$
 - $\lambda z. (\lambda x. x z) z \rightarrow \lambda y. (\lambda x. x y) y$ so
 $\lambda z. (\lambda x. x z) z \sim \lambda y. (\lambda x. x y) y$
 - $\lambda x. (\lambda y. y x) x \sim \lambda y. (\lambda x. x y) y$

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Substitution

- Defined on α -equivalence classes of terms
- $P [N / x]$ means replace every free occurrence of x in P by N
 - P called *redex*; N called *residue*
- Provided that no variable free in P becomes bound in $P [N / x]$
 - Rename bound variables in P to avoid capturing free variables of N

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Substitution

- $x [N / x] = N$
- $y [N / x] = y$ if $y \neq x$
- $(e_1 e_2) [N / x] = ((e_1 [N / x]) (e_2 [N / x]))$
- $(\lambda x. e) [N / x] = (\lambda x. e)$
- $(\lambda y. e) [N / x] = \lambda y. (e [N / x])$ provided $y \neq x$ and y not free in N
 - Rename y in redex if necessary

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Example

$$(\lambda y. y z) [(\lambda x. x y) / z] = ?$$

- Problems?
 - z in redex in scope of y binding
 - y free in the residue
- $(\lambda y. y z) [(\lambda x. x y) / z] \dashrightarrow$
 $(\lambda w. w z) [(\lambda x. x y) / z] =$
 $\lambda w. w (\lambda x. x y)$

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Example

- Only replace free occurrences
- $(\lambda y. y z (\lambda z. z)) [(\lambda x. x) / z] =$
 $\lambda y. y (\lambda x. x) (\lambda z. z)$

Not

$$\lambda y. y (\lambda x. x) (\lambda z. (\lambda x. x))$$

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β reduction

- β Rule: $(\lambda x. P) N \dashrightarrow P [N / x]$
- Essence of computation in the lambda calculus
- Usually defined on α -equivalence classes of terms

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Example

- $(\lambda z. (\lambda x. x y) z) (\lambda y. y z)$
 $\dashrightarrow (\lambda x. x y) (\lambda y. y z)$
 $\dashrightarrow (\lambda y. y z) y \dashrightarrow y z$
- $(\lambda x. x x) (\lambda x. x x)$
 $\dashrightarrow (\lambda x. x x) (\lambda x. x x)$
 $\dashrightarrow (\lambda x. x x) (\lambda x. x x) \dashrightarrow \dots$

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α β Equivalence

- α β equivalence is the smallest congruence containing α equivalence and β reduction
- A term is in *normal form* if no subterm is α equivalent to a term that can be β reduced
- Hard fact (Church-Rosser): if e_1 and e_2 are $\alpha\beta$ -equivalent and both are normal forms, then they are α equivalent

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Order of Evaluation

- Not all terms reduce to normal forms
- Not all reduction strategies will produce a normal form if one exists

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Lazy evaluation:

- Always reduce the left-most application in a top-most series of applications (i.e. Do not perform reduction inside an abstraction)
- Stop when term is not an application, or left-most application is not an application of an abstraction to a term

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Example 1

- $(\lambda z. (\lambda x. x)) ((\lambda y. y y) (\lambda y. y y))$
- Lazy evaluation:
- Reduce the left-most application:
- $(\lambda z. (\lambda x. x)) ((\lambda y. y y) (\lambda y. y y))$
-- β --> $(\lambda x. x)$

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Eager evaluation

- (Eagerly) reduce left of top application to an abstraction
- Then (eagerly) reduce argument
- Then β -reduce the application

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Example 1

- $(\lambda z. (\lambda x. x))((\lambda y. y y) (\lambda y. y y))$
- Eager evaluation:
- Reduce the rator of the top-most application to an abstraction: Done.
- Reduce the argument:
- $(\lambda z. (\lambda x. x))((\lambda y. y y) (\lambda y. y y))$
-- β --> $(\lambda z. (\lambda x. x))((\lambda y. y y) (\lambda y. y y))$
-- β --> $(\lambda z. (\lambda x. x))((\lambda y. y y) (\lambda y. y y))...$

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Example 2

- $(\lambda x. x x)((\lambda y. y y) (\lambda z. z))$
- Lazy evaluation:
 $(\lambda x. x x)((\lambda y. y y) (\lambda z. z)) \rightarrow_{\beta}$

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Example 2

- $(\lambda x. x x)((\lambda y. y y) (\lambda z. z))$
- Lazy evaluation:
 $(\lambda x. \boxed{x} \boxed{x})((\lambda y. y y) (\lambda z. z)) \rightarrow_{\beta}$

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Example 2

- $(\lambda x. x x)((\lambda y. y y) (\lambda z. z))$
- Lazy evaluation:
 $(\lambda x. \boxed{x} \boxed{x})((\lambda y. y y) (\lambda z. z)) \rightarrow_{\beta}$
 $\boxed{((\lambda y. y y) (\lambda z. z))} \boxed{((\lambda y. y y) (\lambda z. z))}$

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Example 2

- $(\lambda x. x x)((\lambda y. y y) (\lambda z. z))$
- Lazy evaluation:
 $(\lambda x. x x)((\lambda y. y y) (\lambda z. z)) \rightarrow_{\beta}$
 $\boxed{((\lambda y. y y) (\lambda z. z))} \boxed{((\lambda y. y y) (\lambda z. z))}$

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Example 2

- $(\lambda x. x x)((\lambda y. y y) (\lambda z. z))$
- Lazy evaluation:
 $(\lambda x. x x)((\lambda y. y y) (\lambda z. z)) \rightarrow_{\beta}$
 $\boxed{((\lambda y. \boxed{y} \boxed{y}) (\lambda z. z))} \boxed{((\lambda y. y y) (\lambda z. z))}$

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Example 2

- $(\lambda x. x x)((\lambda y. y y) (\lambda z. z))$
- Lazy evaluation:
 $(\lambda x. x x)((\lambda y. y y) (\lambda z. z)) \rightarrow_{\beta}$
 $\boxed{((\lambda y. \boxed{y} \boxed{y}) (\lambda z. z))} \boxed{((\lambda y. y y) (\lambda z. z))}$
 $\rightarrow_{\beta} \boxed{((\lambda z. z))} \boxed{((\lambda z. z))} \boxed{((\lambda y. y y) (\lambda z. z))}$

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Example 2

- $(\lambda x. x x)(\lambda y. y y) (\lambda z. z)$
- Lazy evaluation:
 $(\lambda x. x x)(\lambda y. y y) (\lambda z. z) \rightarrow (\lambda y. y y) (\lambda z. z) ((\lambda y. y y) (\lambda z. z))$
 $\rightarrow ((\lambda z. z) (\lambda z. z)) ((\lambda y. y y) (\lambda z. z))$

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Example 2

- $(\lambda x. x x)(\lambda y. y y) (\lambda z. z)$
- Lazy evaluation:
 $(\lambda x. x x)(\lambda y. y y) (\lambda z. z) \rightarrow ((\lambda y. y y) (\lambda z. z)) ((\lambda y. y y) (\lambda z. z))$
 $\rightarrow ((\lambda z. z) (\lambda z. z)) ((\lambda y. y y) (\lambda z. z))$

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Example 2

- $(\lambda x. x x)(\lambda y. y y) (\lambda z. z)$
- Lazy evaluation:
 $(\lambda x. x x)(\lambda y. y y) (\lambda z. z) \rightarrow ((\lambda y. y y) (\lambda z. z)) ((\lambda y. y y) (\lambda z. z))$
 $\rightarrow ((\lambda z. z) (\lambda z. z)) ((\lambda y. y y) (\lambda z. z))$
 $\rightarrow (\lambda z. z) ((\lambda y. y y) (\lambda z. z))$

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Example 2

- $(\lambda x. x x)(\lambda y. y y) (\lambda z. z)$
- Lazy evaluation:
 $(\lambda x. x x)(\lambda y. y y) (\lambda z. z) \rightarrow ((\lambda y. y y) (\lambda z. z)) ((\lambda y. y y) (\lambda z. z))$
 $\rightarrow ((\lambda z. z) (\lambda z. z)) ((\lambda y. y y) (\lambda z. z))$
 $\rightarrow (\lambda z. z) ((\lambda y. y y) (\lambda z. z))$

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Example 2

- $(\lambda x. x x)(\lambda y. y y) (\lambda z. z)$
- Lazy evaluation:
 $(\lambda x. x x)(\lambda y. y y) (\lambda z. z) \rightarrow ((\lambda y. y y) (\lambda z. z)) ((\lambda y. y y) (\lambda z. z))$
 $\rightarrow ((\lambda z. z) (\lambda z. z)) ((\lambda y. y y) (\lambda z. z))$
 $\rightarrow (\lambda z. z) ((\lambda y. y y) (\lambda z. z))$

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Example 2

- $(\lambda x. x x)(\lambda y. y y) (\lambda z. z)$
- Lazy evaluation:
 $(\lambda x. x x)(\lambda y. y y) (\lambda z. z) \rightarrow ((\lambda y. y y) (\lambda z. z)) ((\lambda y. y y) (\lambda z. z))$
 $\rightarrow ((\lambda z. z) (\lambda z. z)) ((\lambda y. y y) (\lambda z. z))$
 $\rightarrow (\lambda z. z) ((\lambda y. y y) (\lambda z. z)) \sim_{\beta} \lambda z. z$

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Example 2

■ $(\lambda x. x x)((\lambda y. y y) (\lambda z. z))$

■ Eager evaluation:

$(\lambda x. x x) \boxed{((\lambda y. y y) (\lambda z. z))} \xrightarrow{\beta}$

$(\lambda x. x x) \boxed{((\lambda z. z) (\lambda z. z))} \xrightarrow{\beta}$

$(\lambda x. x x) \boxed{(\lambda z. z)} \xrightarrow{\beta}$

$(\lambda z. z) (\lambda z. z) \xrightarrow{\beta} \lambda z. z$

η (Eta) Reduction

- η Rule: $\lambda x. f x \xrightarrow{\eta} f$ if x not free in f
 - Can be useful in each direction
 - Not valid in Ocaml
 - recall lambda-lifting and side effects
 - Not equivalent to $(\lambda x. f) x \xrightarrow{\beta} f$ (inst of β)
- Example: $\lambda x. (\lambda y. y) x \xrightarrow{\eta} \lambda y. y$