Problem Set 0

Fall 11

Due: Tuesday, September 6th, in class, before class begins.

Please follow the homework format guidelines posted on the class web page:

`http://www.cs.illinois.edu/class/fa11/cs373/`

You must work on this problem set alone (groups are not allowed).
Consult the following:

- Sipser Chapter 0.
- Chapters 1,2,3 of Discrete Mathematics by Chen (at `http://rutherglen.science.mq.edu.au/wchen/lndmfolder/lndm.html`)
- Lecture on Thursday, August 25th.

1. [**Category**: Notation, **Points**: 20]

   Answer each of the following with "**true**", "**false**" or "**wrong notation**." Follow the notations in Sipser. $\{\ldots\}$ is used to represent sets and not multisets or anything else.

   a) $\{a, b\} \cup \{a, b, c, d\} = \{a, b, c, d\}$

   b) $\{a, b\} \cap \{c, d\} = \varnothing$

   c) $\{a, b\} \cap \{c, d\} = \{\varnothing\}$

   d) $\{a, b\} \setminus \{b, c\} = \{a, c\}$

   e) $\{a, b\}/\{b, c\} = \{a\}$

   f) $\varnothing \in \{a, b, c\}$

   g) $\varnothing \subseteq \{a, b, c\}$

   h) $\varnothing \in \varnothing$

   i) $\varnothing \subseteq \varnothing$

   j) $\{\} = \{\varnothing\}$

   k) $\{a\} \subseteq \{\varnothing, a, b\}$

   l) $\{a\} \subseteq \{a, \{a\}, \{\}\}$

   m) $a \subseteq \{a, \{a\}, \{\}\}$

   n) $\{a\} \in \{a, \{a\}, \{\}\}$

   o) $a \in \{a, b, \{\}\}$

   p) $\{a\} \times \{b, c\} = \{(b, a), (c, a)\}$

   q) $\{a\} \times \{b, c\} = \{a, b\} \times \{c\}$

   r) $|\{a, b\} \times \{c, d\}| = 4$

s) $|powerset(\{a, b, c\})| = 3$

t) $\left|\{a, b\}^3\right| = 3$

Solution:

a) $\{a, b\} \cup \{a, b, c, d\} = \{a, b, c, d\}$ **true**

b) $\{a, b\} \cap \{c, d\} = \varnothing$ **true**

c) $\{a, b\} \cap \{c, d\} = \{\varnothing\}$ **false**

d) $\{a, b\} \setminus \{b, c\} = \{a, c\}$ **false**

e) $\{a, b\}/\{b, c\} = \{a\}$ **wrong notation**; however, everyone gets points for this.

f) $\varnothing \in \{a, b, c\}$ **false**

g) $\varnothing \subseteq \{a, b, c\}$ **true**

h) $\varnothing \in \varnothing$ **false**

i) $\varnothing \subseteq \varnothing$ **true**

j) $\{\} = \{\varnothing\}$ **false**

k) $\{a\} \subseteq \{\varnothing, a, b\}$ **true**

l) $\{a\} \subseteq \{a, \{a\}, \{\}\}$ **true**

m) $a \subseteq \{a, \{a\}, \{\}\}$ **wrong notation**

n) $\{a\} \in \{a, \{a\}, \{\}\}$ **true**

o) $a \in \{a, b, \{\}\}$ **true**

p) $\{a\} \times \{b, c\} = \{(b, a), (c, a)\}$ **false**

q) $\{a\} \times \{b, c\} = \{a, b\} \times \{c\}$ **false**

r) $|\{a, b\} \times \{c, d\}| = 4$ **true**

s) $|powerset(\{a, b, c\})| = 3$ **false**

t) $\left|\{a, b\}^3\right| = 3$ **false**

2. [**Category**: Relations, **Points**: 20]

Let $A = \{a, b, c\}$. Answer each of the following statements about relation on a set A with **true** or **false**. Explain your answer.

a) The relation $R = \{(a, b), (b, a)\}$ is symmetric and reflexive.

b) The relation $R = \{(a, a), (b, b), (c, c)\}$ is not symmetric and reflexive.

c) The relation $R = \{(a, b), (b, c), (c, a)\}$ is symmetric and transitive.

d) The relation $R = \{(a, b), (a, c), (a, a)\}$ is reflexive and transitive.

e) The relation $R = \{(a,a),(b,b),(c,c)\}$ is symmetric, reflexive, and transitive.

Solution:

a) The relation $R = \{(a,b),(b,a)\}$ is symmetric and reflexive. **false, it's not reflexive**

b) The relation $R = \{(a,a),(b,b),(c,c)\}$ is not symmetric and reflexive. **false, it's symmetric**

c) The relation $R = \{(a,b),(b,c),(c,a)\}$ is symmetric and transitive. **false, it's not symmetric**

d) The relation $R = \{(a,b),(a,c),(a,a)\}$ is reflexive and transitive. **false, it's not reflexive**

e) The relation $R = \{(a,a),(b,b),(c,c)\}$ is symmetric, reflexive, and transitive. **true**

Identify whether the following relations are equivalence relations. If not, state one property of equivalence relations that does not hold (reflexive, symmetric, transitive).

(f) $R = \{(a,b) : a = b\}$ on the set $\mathbb{N}$.

(g) $R = \{(a,b) : a < b\}$ on the set $\mathbb{N}$.

(h) $R = \{(a,b) : a - b \text{ } is \text{ } even\}$ on the set $\mathbb{N}$.

(i) $R = \{(a,b) : a + b \text{ } is \text{ } odd\}$ on the set $\mathbb{N}$.

(j) $R = \{(a,b) : a^2 = b^2\}$ on the set $\mathbb{Z}$.

Solution:

(f) $R = \{(a,b) : a = b\}$ on the set $\mathbb{N}$. **true**

(g) $R = \{(a,b) : a < b\}$ on the set $\mathbb{N}$. **false, not symmetric, not reflexive**

(h) $R = \{(a,b) : a - b \text{ } is \text{ } even\}$ on the set $\mathbb{N}$. **true,** $\{2k\}, \{2k-1\} \forall k \in \mathbb{N}$

(i) $R = \{(a,b) : a + b \text{ } is \text{ } odd\}$ on the set $\mathbb{N}$. **false, not reflexive**

(j) $R = \{(a,b) : a^2 = b^2\}$ on the set $\mathbb{Z}$. **true**

3. [**Category**: Functions, **Points**: 10]

Determine whether each of the following functions is one-to-one or onto or both or neither. (Answers could be of the form "one-to-one but not onto" or "neither one-to-one nor onto", etc.)

a) $f_1 : \mathbb{N} \to \mathbb{N} : x \mapsto x + 1$

b) $f_2 : \mathbb{Z} \to \mathbb{Z} : x \mapsto 2x + 1$

c) $f_3 : \mathbb{N} \to \mathbb{N} : x \mapsto x^2$

d) $f_4 : \mathbb{Z} \to \mathbb{Z} : x \mapsto x^2$

e) $f_5 : \mathbb{R} \to \mathbb{Z} : x \mapsto \lfloor x \rfloor$

Solution:

(a) one-to-one, not onto

(b) one-to-one, not onto

(c) one-to-one, not onto

(d) not one-to-one, not onto

(e) not one-to-one, onto

4. [**Category**: Logic, **Points**: 20]

Let $\Sigma = \{a, b, \ldots, z\}$. A finite sequence of symbols chosen from the alphabet Σ can be formally defined as a partial function $pos : \mathbb{N} \to \Sigma$, such that $pos(n)$ is defined if and only if n does not exceed the length of the sequence. For example, the sequence *abc* can be defined as $pos(1) = a$, $pos(2) = b$, $pos(3) = c$, and $pos(n) = \texttt{undef}$ for all $n > 3$.

If we write a first order logic (FOL) sentence over the above notations, it formally describes a property of finite sequences. For example, the sentence $pos(5) = \texttt{undef}$ says that the 5th position of the word is undefined, and hence the length of the sequence is less than 5. Also, the sentence "$\exists n \in \mathbb{N}(pos(n) = a \vee pos(n) = b \vee pos(n) = \texttt{undef})$" says that the sequence is entirely made up of *a*' and *b*'s only.

Translate each of the following formal descriptions into an informal description as simple/natural as possible, e.g., using a short English sentence.

a) $pos(3) \neq \texttt{undef} \wedge pos(4) = \texttt{undef}$

b) $\forall n \in \mathbb{N} \,.\, pos(n) \neq a$

c) $\exists n \in \mathbb{N} \,.\, pos(n) = a \wedge pos(n+1) = \texttt{undef}$

d) $\nexists n \in \mathbb{N} \,.\, pos(n) = a \wedge pos(n+1) = a$

e) $\forall n \in \mathbb{N} \,.\, pos(n) = a \to pos(n+1) = b$

Solution:

a) "the length of the sequence is 3."

b) "there is no 'a' in the sequence."

c) "the last character in the sequence is 'a'."

d) "there is no subsequence 'aa'."

e) "Every 'a' in the sequence is followed by a 'b'."

Translate each of the following English statements into a first-order logic sentence.

(f) "the length of the sequence is at least 7."

(g) "the sequence consists of purely a's."

(h) "the sequence starts with 'ab'."

(i) "every odd position in the sequence is an 'a'."

(j) "the sequence contains a contiguous subsequence of the form '$a?c$', where '?' indicates any single character."

Solution:

a) $pos(7) \neq \texttt{undef}$

b) $\forall n \in \mathbb{N} \,.\, pos(n) = a \vee pos(n) = \texttt{undef}$

c) $pos(1) = a \wedge pos(2) = b$

d) $\forall n \in \mathbb{N} \,.\, pos(2n-1) = a \vee pos(2n-1) = \texttt{undef}$

e) $\exists n \in \mathbb{N} \,.\, pos(n) = a \wedge pos(n+2) = c$