University of Illinois at Urbana-Champaign
Department of Computer Science

Midterm 1

CS 373: Theory of Computation
Fall 2009

Name:
Netid:

- Print your name and netid, *neatly* in the space provided above; print your name at the upper right corner of *every* page. Please print legibly.
- This is a *closed book* exam. No notes, books, dictionaries, calculators, or laptops are permitted.
- You are free to cite and use any theorems from class or homeworks without having to prove them again.
- Write your answers in the space provided for the corresponding problem. Let us know if you need more paper.
- Suggestions: Read through the entire exam first before starting work. Do not spend too much time on any single problem. If you get stuck, move on to something else and come back later.
- If you run short on time, remember that partial credit will be given.
- If any question is unclear, ask us for clarification.

Question	Points	Score
Problem 1	20	
Problem 2	15	
Problem 3	15	
Problem 4	15	
Problem 5	15	
Problem 6	20	
Total	100	

1. Short Problems (20 points)

   Give answers to each of the following questions, including a short justification. For each, you will get two points for the correct answer and two points for the correct justification.

   (a) For a language A and DFA $M = (Q, \Sigma, \delta, q_0, F)$ for which $A = L(M)$, what can be inferred about the number of equivalence classes for each of the equivalence relations $\sim_A$ and $\sim_M$? (4 points)

   (b) If A' is a nonregular language and $A' \subset A$, then must A be nonregular? (4 points)

   (c) Is $\{(jimmy)^n(john)^n \in \Sigma^* \mid n \geq 0\}$ a regular language over the English alphabet? (4 points)

   (d) For any regular expression R, is it always true that $(R^*R \cup R)^* = R^*$? (4 points)

   (e) If a regular language is infinite, then does every DFA that recognizes it contain cycles (when the DFA drawn as a directed graph)? (4 points)

   Solution:

   (a) number of classes with respect to $\sim_A \leq$ number of classes with respect to $\sim_M$.

   (b) No, for example $\{0^n1^n \mid n \geq 0\} \subseteq \{0,1\}^*$.

   (c) No, because if it is, then it should remain regular under the homomorphism which maps i to 0 and h to 1 and every other thing to ϵ. Note that after this homorphism we have the set $\{0^n1^n \mid n \geq 0\}$ which we know that it is not regular.

   (d) Yes. Since $\epsilon \in R^*$, we have $R \subseteq R^*R$, so $R^*R \cup R = R^*R = R^+$. And note that by definition $(R^+)^* = R^*$, since both these two sets contain exactly all strings that are concatenations of a few elements of R.

   (e) Yes; if the DFA has m states, then consider a string in the language with more than m characters. The path of acceptance for this string must have a cycle in the DFA because some state on this path should appear at least twice.

2. Induction (15 points)

   Prove by induction that for any two langauges A and B, if $A \subseteq B$ then $A^n \subseteq B^n$ for all $n \geq 1$. Here, $A^2 = AA$, $A^3 = AAA$, and so on.

   Solution:

   We use induction on n to prove that $A^n \subseteq B^n$.

   As for the <u>base case</u>, $n = 1$, we need to show that $A \subseteq B$, and we see that this fact is given as an assumption, so this part is done.

   Now for the <u>inductive step</u>, we assume that for some arbitrary $k \geq 1$, we have $A^k \subseteq B^k$ (i.e. <u>induction hypothesis</u>), then using that we prove $A^{k+1} \subseteq B^{k+1}$: Let $x \in A^{k+1}$. By definition $x = yz$ where $y \in A^k$ and $z \in A$. By induction hypothesis $A^k \subseteq B^k$ and so $y \in B^k$. It is given that $A \subseteq B$, therefore $z \in B$. By definition, since $y \in B^k$ and $z \in B$, we have $x = yz \in B^{k+1}$. So we have proved that every member of A^{k+1} appears in B^{k+1}, which means $A^{k+1} \subseteq B^{k+1}$.

   This complete the induction, which proves that $A^n \subseteq B^n$ for $n \geq 1$, if $A \subseteq B$.

3. Closure Properties (15 points)

   Use closure properties to prove that

$$A = \{w \in \{0,1\}^* \mid \text{ the number of 0s in } w \text{ differs from the number of 1s } \}$$

   is nonregular.

   Solution: Assume to the contrary that A is regular. By closure properties, $\overline{A}$ is regular. Now again by closure properties $\overline{A} \cap L(0^*1^*)$ is regular. Note that

$$\overline{A} \cap L(0^*1^*) = \{0^n 1^n \mid n \geq 0\}$$

   which we know is not a regular language and therefore we have reached a contradiction. This means that A can not be regular.

4. Pumping Lemma (15 points)

   Use the pumping lemma to prove that

$$A = \{(ba)^n b^n \in \{a, b\}^n \mid n \geq 0\}$$

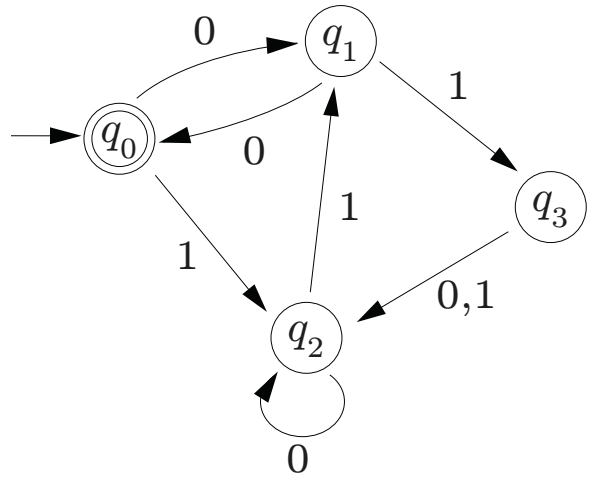
   is nonregular.

   Solution:

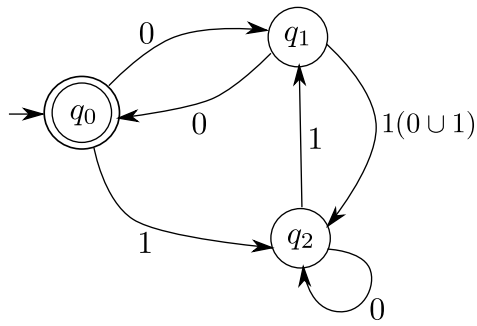
   Assume to the contrary that A is regular. Therefore it has a pumping length $p \geq 1$. Consider $s = (ba)^p b^p$, we observe that $s \in A$. Since $|s| \geq p$, <u>every proper</u> break down of it must be "pumpable". Let $s = xyz$ be a proper break down of string x according to pumping lemma, that is $|xy| \leq p$ and $y \neq \epsilon$. Since $|(ba)^p| = 2p$, the string xy is just a prefix of $(ba)^p$. Since $y \neq \epsilon$, when we pump down y, we will remove at least one a or one b from the $(ba)^p$ prefix of s while s will have still p trailing b's; this means that $xz \notin A$, which contradicts the pumping lemma. Thus A can not be regular.

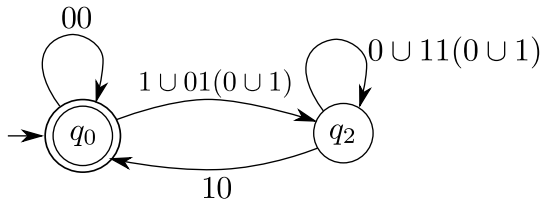
5. DFA to Regular Expression (15 points)

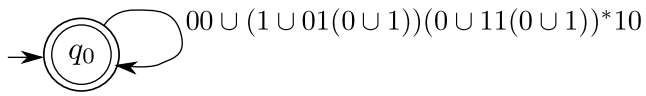
Convert the following DFA into a regular expression that describes the same language. Show your work.



Solution: We eliminate states in the order q_3, q_1, q_2. We opt not to add dummy intial and final states here, since they don't play a role in this sample (there is just one final state in the DFA).







So our regular expression is $\Big(00 \cup (1 \cup 01(0 \cup 1))(0 \cup 11(0 \cup 1))^*10\Big)^*$.

6. Regularity (20 points)

   (a) Prove that
   $$A = \{0^n 1^k \in \{0,1\}^* \mid n \geq 0 \text{ and } k \leq 5\}$$
   is regular. (15 points)

   (b) Determine the Nerode equivalence classes (induced by $\sim_A$). You do not need to justify your work for this part. (5 points)

Solution:

(a) A is the language of a regular expression, $A = L\Big(0^*(\epsilon \cup 1 \cup 11 \cup 111 \cup 1111 \cup 11111)\Big)$.

(b) $\mathcal{C}_1 = L(0^*),\ \mathcal{C}_2 = L(0^*1),\ \mathcal{C}_3 = L(0^*11),\ \mathcal{C}_4 = L(0^*111)$
$\mathcal{C}_5 = L(0^*1111),\ \mathcal{C}_6 = L(0^*11111),\ \mathcal{C}_7 = L((0+1)^*) - \mathcal{C}_1 - \mathcal{C}_2 - \mathcal{C}_3 - \mathcal{C}_4 - \mathcal{C}_5 - \mathcal{C}_6$.