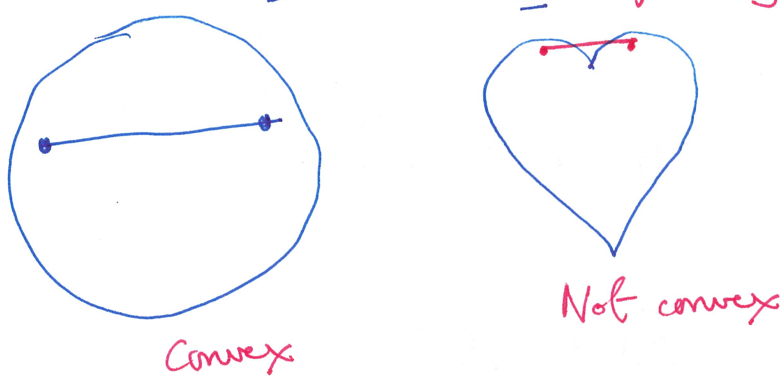
Lecture 12 (CS 307)

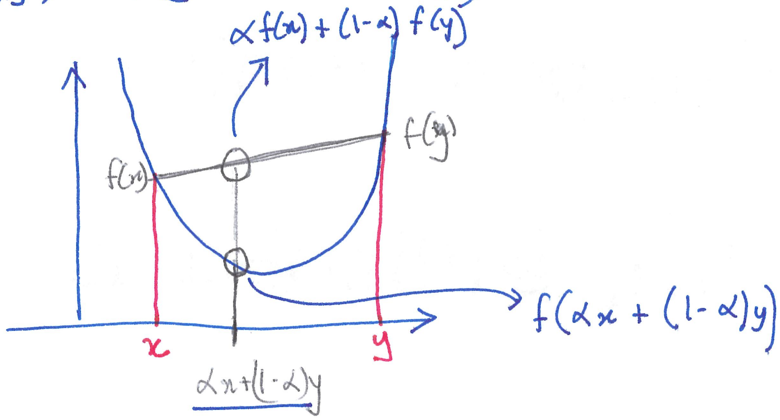
Convex Set: A set of points $C \subseteq \mathbb{R}^d$ is said to be convex if $\forall x, y \in C$ then all points in the line segment from x to y also belong to C

$$\underbrace{\alpha \underline{x} + (1-\alpha)\underline{y}}_{\text{line segment from } x \text{ to } y} \text{ for any } \alpha \in [0,1].$$



Convex Function: A function $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ is convex iff $\forall x, y \in \mathbb{R}^d$

$$\forall \alpha \in [0,1], \quad f(\alpha x + (1-\alpha) y) \leq \alpha f(x) + (1-\alpha) f(y)$$



Example $C = \{ x \mid \theta^T x \le 0 \}$ (halfspace).

To show C is convex.

$$\underbrace{\forall x, y \in C}, \quad \forall \alpha \in [0,1], \quad \alpha x + (1-\alpha) y \in C.$$

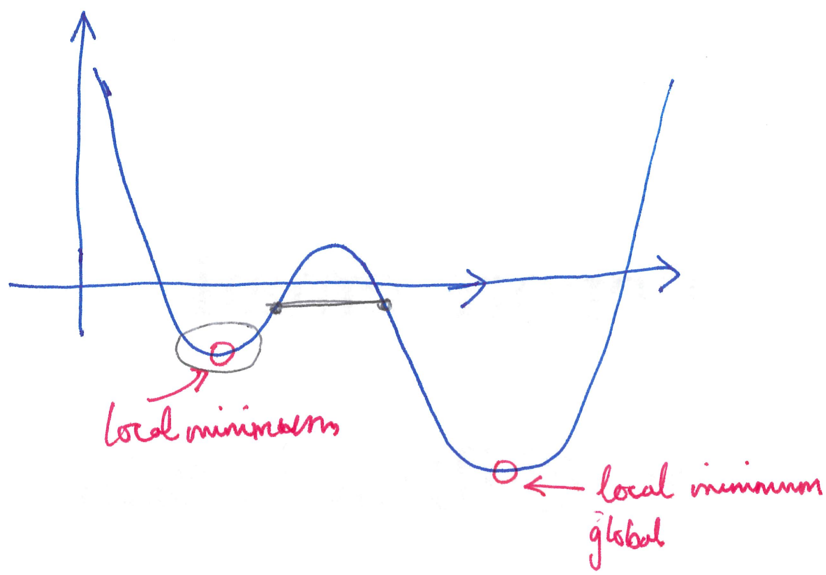
$$\Rightarrow \theta^T x \le 0, \quad \theta^T y \le 0$$

$$\theta^T(\alpha x + (1-\alpha) y) = \underbrace{\overset{\ge 0}{\alpha}\, \overset{\le 0}{\theta^T x}}_{\le 0} + \underbrace{\overset{\ge 0}{(1-\alpha)}\, \overset{\le 0}{\theta^T y}}_{\le 0} \le 0$$

For any function $f : \mathbb{R}^d \to \mathbb{R}$

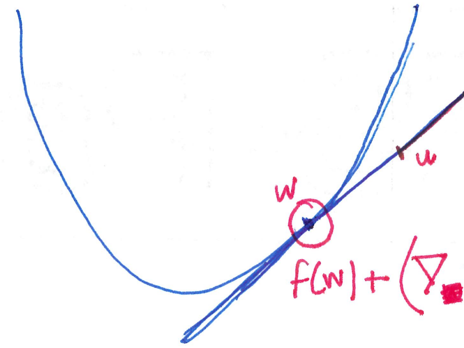
$$\text{epigraph}(f) = \{ (x,y) \mid y \ge f(x) \} \subseteq \mathbb{R}^{d+1}$$

Proposition: f is convex iff epigraph(f) is convex.



Theorem: Every local minimum of a convex function f is a global minimum.

~~Ex~~ If f is convex then there is (a) tangent that lies below f.



$f(w) + (\nabla f(w))^T (u-w)$

$= 0$

When f is "differentiable" (the gradient of f is defined).

$$f(u) \geq f(w) + (\nabla f(w))^T (u-w).$$

$x_1^2 + x_2^2 + \cdots$

Proposition: $f : \mathbb{R} \to \mathbb{R}$ then the following statements are equivalent (when f is twice differentiable).

- f is convex.
- f' is a non-decreasing.
- f'' is positive.

Example: $f(x) = x^2$. $(x \in \mathbb{R})$.

$f'(x) = 2x$, $f''(x) = 2 > 0$. (f is convex).

$$g(x) = \ln(1+e^x)$$

$$g'(x) = \frac{1}{1+e^x} \cdot e^x = \frac{e^x}{1+e^x} = \frac{1}{1+e^{-x}}$$

As x increases, $g'(x)$ increases $\longrightarrow$ g is convex.

Proposition: Suppose $f : \mathbb{R}^d \to \mathbb{R}$ where

$$f(x) = g(\theta^T x + \theta_0) \qquad g : \mathbb{R} \to \mathbb{R}.$$

f is convex if g is convex.

Proposition: $f_i : \mathbb{R}^d \to \mathbb{R}$ for $i \in \{1, \dots n\}$. such f_i is convex.

(a) $g(x) = \max\limits_{i \in \{1, \dots n\}} f_i(x)$. is convex.

(b) $h(x) = \alpha_1 f_1(x) + \alpha_2 f_2(x) + \dots + \alpha_n f(n)$. is convex.