CS 196.73: Discrete Structures (Honors), Spring 2013

Assignment 2

Due April 29 in the 173 honors dropbox in the basement of Siebel. Your homework must be formatted with latex, but you should just turn in the hardcopy (not e.g. the latex source). Exception: the code printout for problem 1 does not have to be formatted. Simply staple it to the end of your writeup.

1. Students at Brutal U. would like to find out the highest floor of their 64-floor CS building that they can push a CS professor from, and still have him survive. If they don't want to lose more than one CS professor, then the only strategy is to push him from the 1st floor, the 2nd floor, the 3rd floor, until a floor k is found such that the professor survived the push from floor k, but died when pushed from floor $k+1$. (Or, survived even the push from the 64th floor.) If they push him from a floor f without having tried floor $f-1$, then if he dies they won't be able to determine whether or not he would have survived the fall from floor $f-1$. So, in the worst case, this will take 64 trials (pushes) to determine the maximum floor, using this simple "try each floor successively" strategy.

   However, if there are two professors available, then fewer than 64 trials are needed. For example, the students might try pushing the first professor out floors numbered 1, 2, 4, 8, 16, 32, and 64, until one is found from which he perishes. In the worst case, this will be floor 64. The remaining professor can be used to determine, via a successive-floor strategy, which of the floors 33 through 63 is the maximum survivable floor. This strategy uses, in the worst case, a total of 7+31 = 38 trials. But there are better strategies.

   (a) If using the strategy above, what is the fewest number of trials, the worst case, for using two professors to find the maximum survivable fall in an N-story building? Assume that N is a power of 2.

   (b) Describe a better strategy (the best you can come up with) for minimizing the worst-case number of trials required by two professors. Hint: There is at least one strategy that can solve the 64-floor problem in at most 15 trials.

   (c) Following your strategy, how many trials would be required for $N = 64$ floors in the best case and in the worst case? What is the worst-case number of trials for N floors in general? You can assume a particular form of N (e.g., N must be a power of two or an even number or a square of an integer).

   (d) Write a program that simulates the experiment for 64 floors using both strategies (the one described in the preamble and your proposed strategy). Your program should choose a floor uniformly at random (e.g., "k=ceil(rand(1)*64)" in Matlab) and then apply each strategy to solve for the maximum survivable floor with 2 professors.

Include a printout of your program in your writeup. Make sure your code is easy to read.

(e) Do some simulations with your program:

(i) Run your program 10 times and report the min, mean, and max number of trials for each strategy.

(ii) Run your program 1000 times and report the min, mean, and max number of trials for each strategy.

(iii) Compare these numbers to your estimated best-case and worst-case number of trials for each strategy. Comparison can be done by creating a table that contains your best/worst-case predictions and the results of your simulation for the 10-iteration and 1000-iteration runs.

(f) Describe a good strategy if *three* professors are available. What is the worst-case number of trials required for $N = 64$ floors using this strategy?

2. Suppose we have the numbers 1 through n written in a permuted string. For example, for $n = 6$, we might have "215346". We'd like to sort these numbers into the correct order, from lowest to highest. Assume though that the only operation available to us is to call a procedure "reverse-prefix" which can reverse an initial prefix of the given string. For example, reverse-prefix(215346,3) would result in the string 512346, which is the result of reversing the first 3 characters of 215346. Thus, reverse-prefix takes as input a string s, and a length k, and outputs the results of reversing the prefix of length k in s.

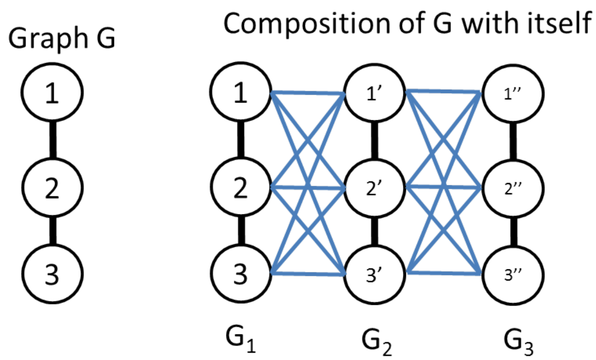
If we apply reverse-prefix again, at position 5, we get reverse-prefix(512346,5) = 432156. Applying it one more time at position 4 gives us reverse-prefix(432156,4) = 123456, and the sequence is in order.

(a) Show that it is always possible to sort a permuted string of n numbers into the correct order using only the reverse-prefix operation. Do this by describing a general method for sorting using the reverse-prefix operation. How many applications of reverse-prefix, in the worst case, does your method take given n numbers?

(b) Prove that at least $n - 1$ applications of reverse-prefix are needed to sort n numbers. Hint: look at the number of "bad adjacencies" which must be separated by a reverse-prefix application. Try to find a string with a lot of bad adjacencies.

3. The composition of two graphs G and H is denoted $G(H)$ and formed as follows. For each node u of G, we create an entire copy of H, which we call H_u. If H_u and H_v represent adjacent nodes u and v in G, then we create an edge between each node of H_u and each node of H_v. (In other words, we create the complete bipartite graph with H_u on one side, and H_v on the other.)

   Below is an example of the composition of a graph with itself.

   Suppose that G is a graph (not necessarily the graph shown above) with chromatic number n. Prove an upper bound on the chromatic number of $G(G)$.