BIOE 298, SECTIONS MFI & B

PRACTICE EXAM 3

You have 80 minutes to complete this exam.
You may use notes or printouts from the course website,
but **no electronic resources**.

Circle your final answer for each question.

Part I (20 points; 4 points each)

(1) Is the vector $\begin{pmatrix} a \\ b \end{pmatrix}$ an eigenvector of the matrix $\begin{pmatrix} 3 & 4/b \\ 0 & 1 \end{pmatrix}$?

$\underline{A}\,\underline{x} = \lambda \underline{x}$

$$\begin{pmatrix} 3 & 4/b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 3a + \frac{4}{\cancel{b}}\cancel{b} \\ 0 + b \end{pmatrix} = \begin{pmatrix} 3a+4 \\ b \end{pmatrix} \neq \lambda \begin{pmatrix} a \\ b \end{pmatrix}$$

No; it is not.

(2) True or False. Let $\mathbf{A} \in \mathbb{R}^{n \times n}$. The matrix $\underline{\mathbf{A}}$ is perfect if the SVD of $\mathbf{A}$ reveals n distinct singular values (the diagonal entries along $\mathbf{\Sigma}$).

n distinct λ_i

($\underline{A}$ is symmetric + p.d., then this is true.)

(3) We showed that quadratic programs can be solved to global optimality if the quadratic objective is convex. Is the quadratic program that results from the Support Vector Machine (SVM) problem solvable to global optimality? Why or why not?

SVM $\min \sum a_i^2$ s.t. ...

quadratic is convex iff $\underline{Q}$ is p.d.

$\underline{x}^T \underline{Q}\, \underline{x} > 0 \;\; \forall\, \underline{x} \neq \underline{0}$

It is globally solvable since $\sum a_i^2 \geq 0 \;\forall\, a_i$.

(4) Give a set of three vectors that span the space $\mathbb{R}^2$.

$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + 0 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Can your set of vectors also serve as a basis for $\mathbb{R}^2$?

No. A basis in $\mathbb{R}^2$ has exactly two vectors.

(5) True or False. Adding principal components to a Principal Components Regression model always increases the accuracy of the model but makes the model more difficult to interpret.

False

True

Part II (30 points)

Construct an orthonormal basis from the vectors $\left\{\begin{pmatrix}3\\4\end{pmatrix}, \begin{pmatrix}0\\-1\end{pmatrix}\right\}$.

$$u_1 = \begin{pmatrix}3\\4\end{pmatrix}$$

$$u_2 = \begin{pmatrix}0\\-1\end{pmatrix} - \underbrace{\text{proj}_{u_1}\begin{pmatrix}0\\-1\end{pmatrix}}_{\frac{\begin{pmatrix}0\\-1\end{pmatrix}\cdot\begin{pmatrix}3\\4\end{pmatrix}}{\begin{pmatrix}3\\4\end{pmatrix}\cdot\begin{pmatrix}3\\4\end{pmatrix}}\begin{pmatrix}3\\4\end{pmatrix} = \frac{-4}{25}\begin{pmatrix}3\\4\end{pmatrix}}$$

$$= \begin{pmatrix}0\\-1\end{pmatrix} + \frac{4}{25}\begin{pmatrix}3\\4\end{pmatrix} = \frac{25}{25}\begin{pmatrix}0\\-1\end{pmatrix} + \frac{4}{25}\begin{pmatrix}3\\4\end{pmatrix}$$

$$= \begin{pmatrix}12/25\\-9/25\end{pmatrix}$$

$$\|\underline{u}_2\| = \sqrt{\left(\frac{12}{25}\right)^2 + \left(\frac{-9}{25}\right)^2}$$

$$= \frac{1}{25}\sqrt{12^2+9^2}$$

$$= 15/25 = 3/5$$

$$\boxed{\hat{u}_1 = \begin{pmatrix}3/5\\4/5\end{pmatrix}, \quad \hat{u}_2 = \begin{pmatrix}4/5\\-3/5\end{pmatrix}}$$

Decompose the vector $\begin{pmatrix}1\\-1\end{pmatrix}$ onto your basis.

$$a_1 = \begin{pmatrix}1\\-1\end{pmatrix}\cdot\begin{pmatrix}3/5\\4/5\end{pmatrix} = -\frac{1}{5}$$

$$a_2 = \begin{pmatrix}1\\-1\end{pmatrix}\cdot\begin{pmatrix}4/5\\-3/5\end{pmatrix} = \frac{7}{5}$$

$$\boxed{\begin{pmatrix}1\\-1\end{pmatrix} = -\frac{1}{5}\begin{pmatrix}3/5\\4/5\end{pmatrix} + \frac{7}{5}\begin{pmatrix}4/5\\-3/5\end{pmatrix}}$$

Part III (30 points)

Convert the following system of ODEs into the matrix form $\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$, $\mathbf{x}(0) = \mathbf{x}_0$.

$$\frac{dx_1}{dt} = -x_1 + 2x_2$$
$$\frac{dx_2}{dt} = 3x_1 + 4x_2$$
$$x_1(0) = 0, \quad x_2(0) = 1$$

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\frac{d\underline{x}}{dt} = \underbrace{\begin{pmatrix} -1 & 2 \\ 3 & 4 \end{pmatrix}}_{\underline{A}} \underline{x}, \quad \underline{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

If the eigenvalues and eigenvectors of $\mathbf{A}$ are

$$\lambda_1 = -2, \mathbf{v_1} = \begin{pmatrix} -0.9 \\ 0.4 \end{pmatrix}, \quad \lambda_2 = 5, \mathbf{v_2} = \begin{pmatrix} -0.3 \\ -1 \end{pmatrix}$$

what is the solution to the systems of ODEs?

$$\underline{x}(t) = c_1 \underline{v}_1 e^{\lambda_1 t} + c_2 \underline{v}_2 e^{\lambda_2 t} = \boxed{\frac{5}{17}\begin{pmatrix} -0.9 \\ 0.4 \end{pmatrix} e^{-2t} - \frac{15}{17}\begin{pmatrix} -0.3 \\ -1 \end{pmatrix} e^{5t}}$$

$$(\underline{v}_1 \ \underline{v}_2)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}$$

$$\begin{pmatrix} -0.9 & -0.3 \\ 0.4 & -1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} -0.9 & -0.3 & 0 \\ 0.4 & -1 & 1 \end{pmatrix}$$

$$\xrightarrow{-\frac{1}{0.3}R_1} \begin{pmatrix} 3 & 1 & 0 \\ 2/5 & -1 & 1 \end{pmatrix}$$

$$\xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 2/5 & -1 & 1 \\ 3 & 1 & 0 \end{pmatrix}$$

$$\xrightarrow{R_2 + 12R_1} \begin{pmatrix} 2/5 & -1 & 1 \\ 17/5 & 0 & 1 \end{pmatrix} \Rightarrow \begin{array}{l} 17/5\, c_1 = 1 \Rightarrow c_1 = 5/17 \\ \frac{2}{\cancel{5}}\frac{\cancel{5}}{17} - c_2 = 1 \Rightarrow c_2 = -15/17 \end{array}$$

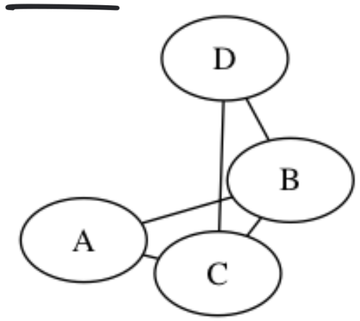
Is the system stable? Why or why not?

No. Since $\lambda_2 > 0$, $\underline{x}(t) \to \begin{pmatrix} \infty \\ \infty \end{pmatrix}$ as $t \to \infty$.

Part IV (20 points)

Define an adjacency matrix **A** for the following four-node, undirected network.

$$\underline{A} = \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \end{array} \begin{array}{c} A \\ B \\ C \\ D \end{array}$$



Given

~~Give~~ the following output from Matlab, report the most "central" node in the network and its centrality score.

```
>> [V,L] = eig(A)

V =

   -0.5573   -0.0000    0.7071   -0.4352
    0.4352    0.7071    0.0000   -0.5573
    0.4352   -0.7071   -0.0000   -0.5573
   -0.5573    0.0000   -0.7071   -0.4352


L =

   -1.5616         0         0         0
         0   -1.0000         0         0
         0         0    0.0000         0
         0         0         0    2.5616
```

largest magnitude

λ_{max}

B & C are both equally the most central.