PROBLEMS 19

Construct an orthonormal basis for $\mathbb{R}^3$ from

$$\underline{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \underline{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \underline{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

First, we note that $\underline{v}_1, \underline{v}_2, \underline{v}_3$ are not already orthonormal.

$$\|\underline{v}_1\| = \sqrt{2} \neq 1$$

$$\|\underline{v}_2\| = \sqrt{3} \neq 1$$

$$\|\underline{v}_3\| = \sqrt{2} \neq 1$$

Not <u>Normalized</u>

$$\underline{v}_1 \cdot \underline{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 2 \neq 0$$

$$\underline{v}_1 \cdot \underline{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = 1 \neq 0$$

$$\underline{v}_2 \cdot \underline{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = 2 \neq 0$$

Not <u>orthogonal</u>

Using Gram-Schmidt to find $\hat{u}_1, \hat{u}_2, \hat{u}_3$

$$\underline{u}_1 = \underline{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\text{proj}_{\underline{u}_1}(\underline{v}_2) = \frac{\underline{u}_1 \cdot \underline{v}_2}{\underline{u}_1 \cdot \underline{u}_1} \underline{u}_1$$

$$\underline{u}_2 = \underline{v}_2 - \text{proj}_{\underline{u}_1}(\underline{v}_2)$$

$$= \frac{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \frac{2}{2}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\underline{u}_3 = \underline{v}_3 - \underbrace{\text{proj}_{\underline{u}_1}(\underline{v}_3)} - \underbrace{\text{proj}_{\underline{u}_2}(\underline{v}_3)} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1/2 \\ 1/2 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1/2 \\ 1/2 \\ 0 \end{pmatrix}$$

$$\text{proj}_{\underline{u}_1}(\underline{v}_3) = \frac{\underline{u}_1 \cdot \underline{v}_3}{\underline{u}_1 \cdot \underline{u}_1} \underline{u}_1$$

$$\text{proj}_{\underline{u}_2}(\underline{v}_3) = \frac{\underline{u}_2 \cdot \underline{v}_3}{\underline{u}_2 \cdot \underline{u}_2} \underline{u}_2$$

$$= \frac{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}}{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$= \frac{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}}{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

so $\underline{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\underline{u}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, $\underline{u}_3 = \begin{pmatrix} -1/2 \\ 1/2 \\ 0 \end{pmatrix}$

Check orthogonality:

$$\underline{u}_1 \cdot \underline{u}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0 \quad \checkmark$$

$$\underline{u}_1 \cdot \underline{u}_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -1/2 \\ 1/2 \\ 0 \end{pmatrix} = 1/2 - 1/2 = 0 \quad \checkmark$$

$$\underline{u}_2 \cdot \underline{u}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1/2 \\ -1/2 \\ 0 \end{pmatrix} = 0 \quad \checkmark$$

Now we normalize

$$\hat{u}_1 = \frac{1}{\sqrt{1^2+1^2}} \underline{u}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\hat{u}_3 = \frac{1}{\sqrt{(-1/2)^2 + (1/2)^2}} \underline{u}_3$$

$$= \sqrt{2}\, \underline{u}_3 = \begin{pmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \\ 0 \end{pmatrix}$$

$\hat{u}_2 = \underline{u}_2$ since $\|\underline{u}_2\| = 1$