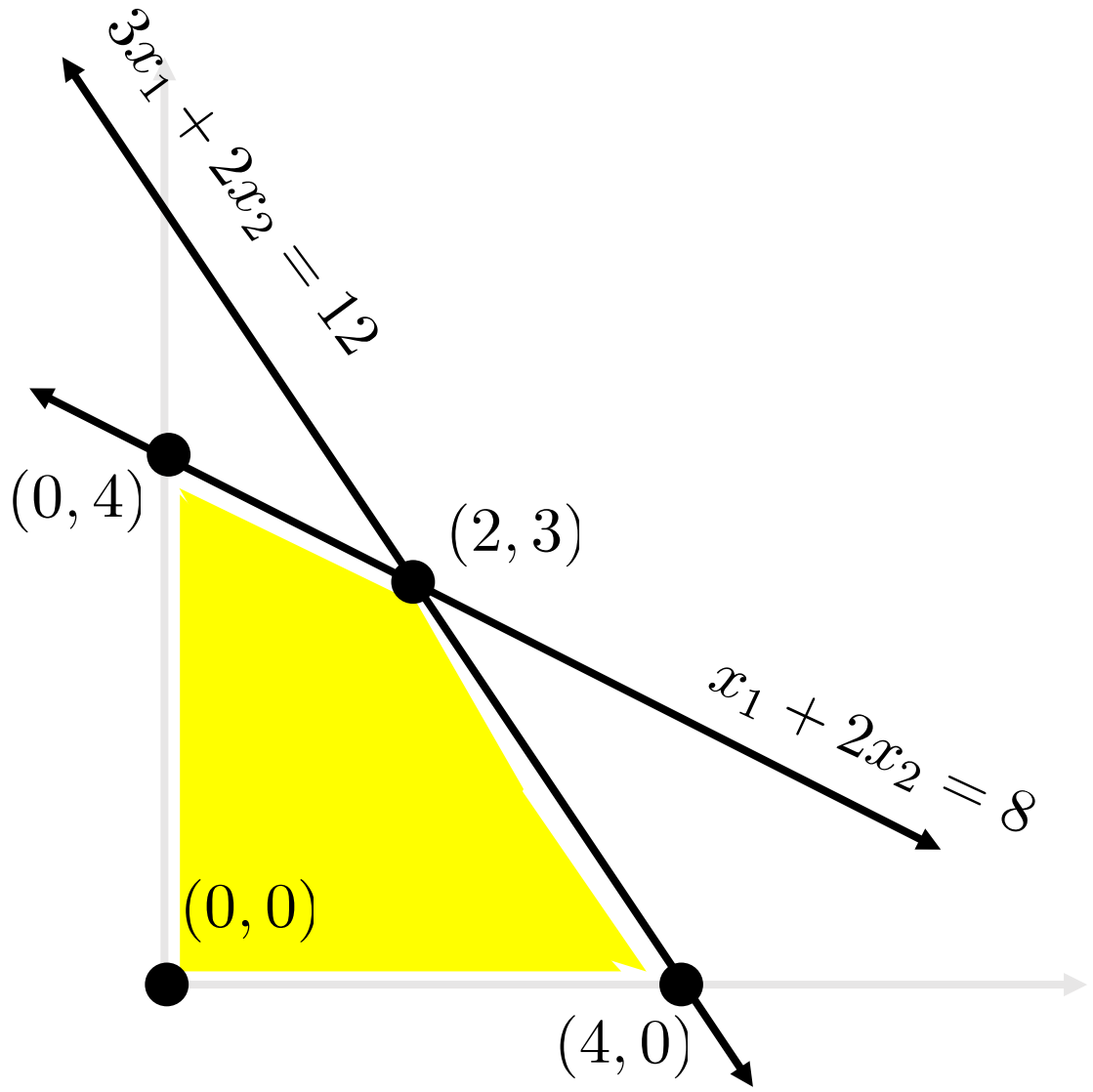
Linear Programming (LP) with inequality constraints define a feasible region.

$$\max\ 2x_1 + 3x_2$$

subject to

$$x_1 + 2x_2 \leq 8$$
$$3x_1 + 2x_2 \leq 12$$
$$x_1,\ x_2 \geq 0$$

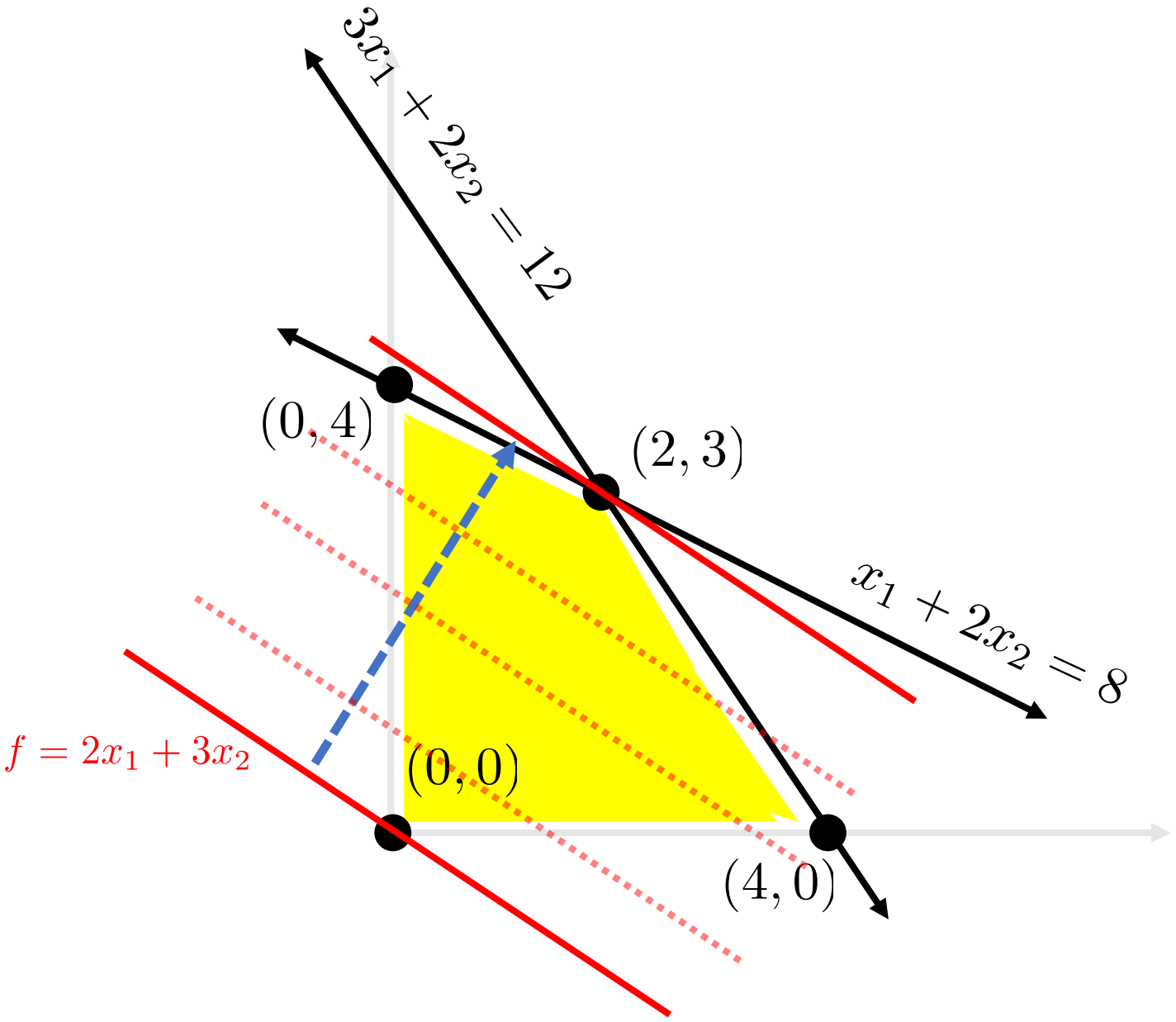


The LP objective function is a hyperplane; optimization moves the plane away (max) from zero.

$$\max\ 2x_1 + 3x_2$$

subject to

$$x_1 + 2x_2 \leq 8$$
$$3x_1 + 2x_2 \leq 12$$
$$x_1,\ x_2 \geq 0$$



Manual Solution by Enumeration

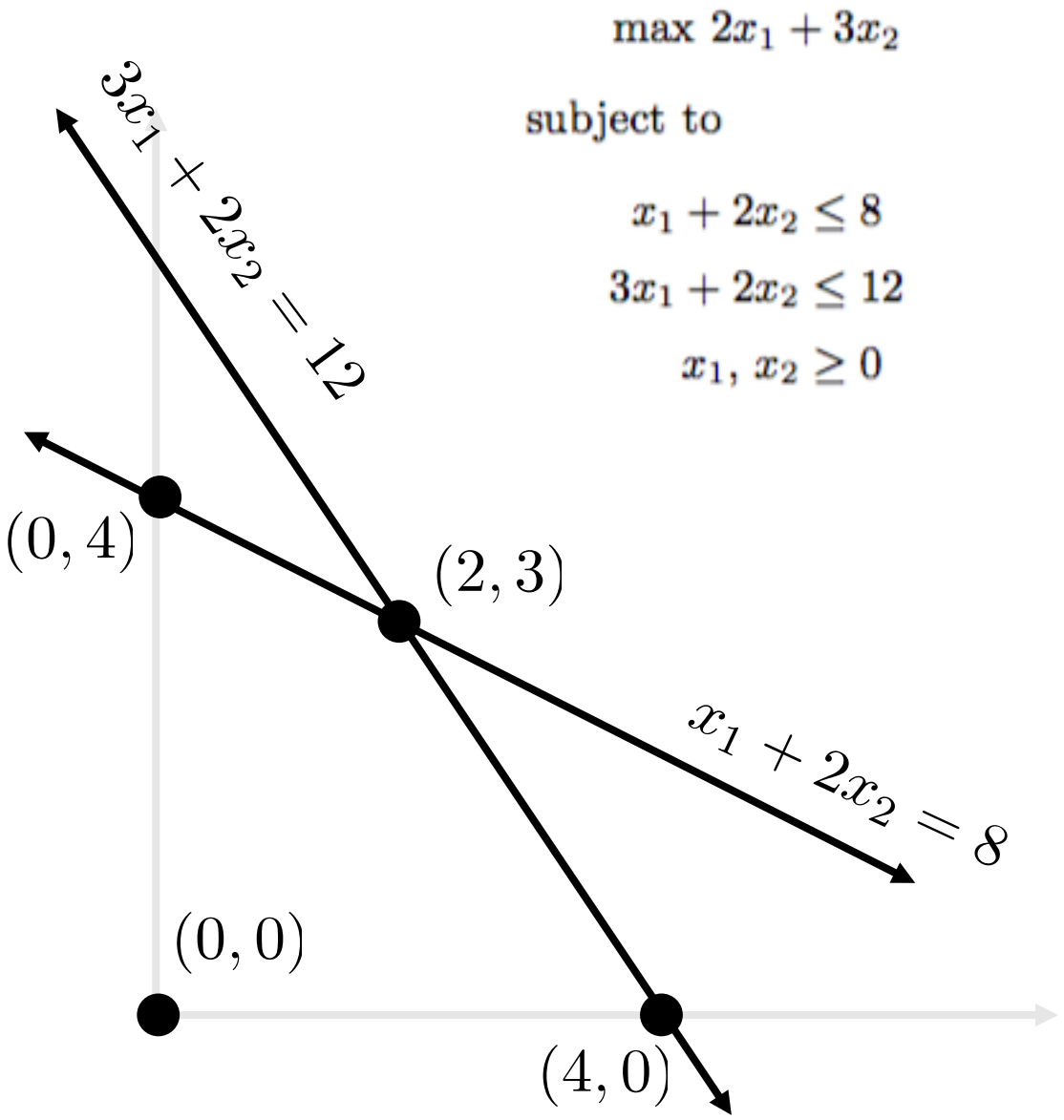
We know the optimal solution lies at a corner point.
In 2D, we can enumerate all corner points.

f(0,0) = 2(0) + 3(0) = 0
f(4,0) = 2(4) + 3(0) = 8
f(0,4) = 2(0) + 3(4) = 12
f(2,3) = 2(2) + 3(3) = 13

Let's say we start at (0,0). So long as we always move to an adjacent corner point with a greater objective value, we will end up at the optimal solution.

(0,0) → (0,4) → (2,3)

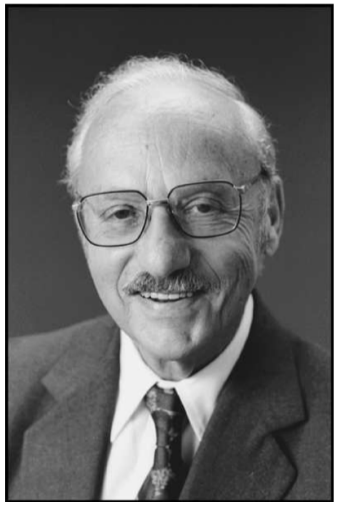
We can skip point (4,0).



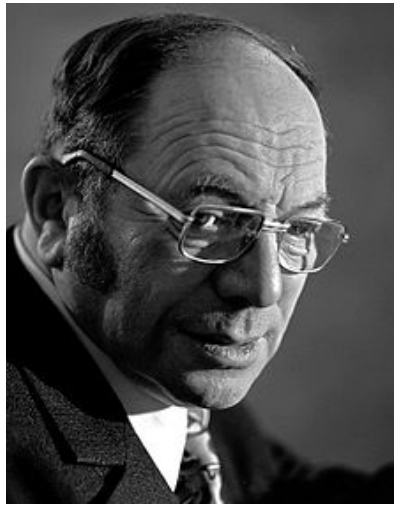
The Simplex Method

Invented by Dantzig in 1947 (or earlier, the US Air Force prevented publication during WWII).

Independently invented by Kantorovich in 1939, but this was unknown outside of the Soviet Union.



George Dantzig



Leonid Kantorovich

Beginning with a feasible corner point, the Simplex Method identifies on those corner points that are

- adjacent to the current corner point
- have a better objective value

The cycle continues until no adjacent corner points improve the objective. This is a local optimum, which is global for an LP.

Step 0: Transform the problem into standard form.

$$\max\ 2x_1 + 3x_2$$

subject to

$$x_1 + 2x_2 \le 8$$
$$3x_1 + 2x_2 \le 12$$
$$x_1,\ x_2 \ge 0$$

$$\max\ f = \mathbf{c}^{\mathrm{T}}\mathbf{x}$$

subject to

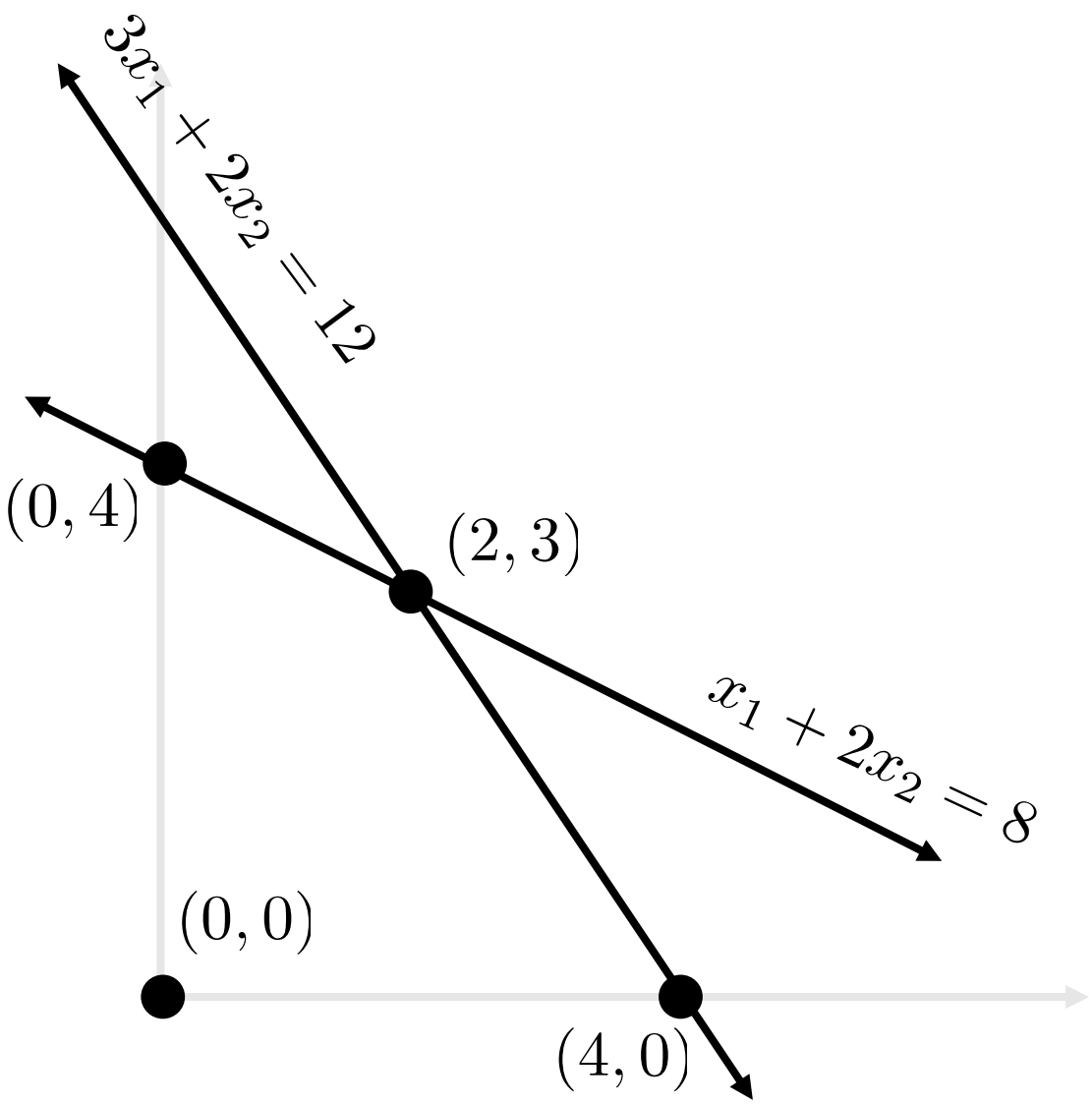
$$\mathbf{A}\mathbf{x} = \mathbf{b}$$
$$\mathbf{x} \ge \mathbf{0}$$

$$\max\ 2x_1 + 3x_2$$

subject to

$$x_1 + 2x_2 + s_1 = 8$$
$$3x_1 + 2x_2 + s_2 = 12$$
$$x_1,\ x_2,\ s_1,\ s_2 \ge 0$$

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \end{pmatrix} \quad \mathbf{c}^{\mathrm{T}} = \begin{pmatrix} 2 \\ 3 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{A} = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 3 & 2 & 0 & 1 \end{pmatrix} \mathbf{b} = \begin{pmatrix} 8 \\ 12 \end{pmatrix}$$



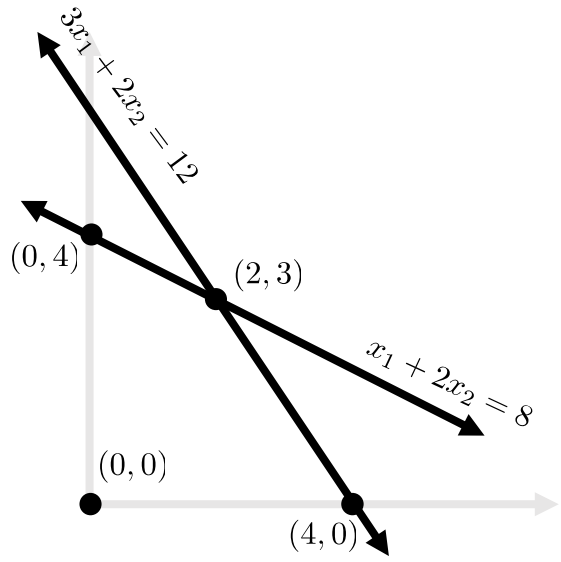
Step 1: Set up Initial Tableau

$$\max\ 2x_1 + 3x_2$$

subject to

$$x_1 + 2x_2 + s_1 = 8$$
$$3x_1 + 2x_2 + s_2 = 12$$
$$x_1,\ x_2,\ s_1,\ s_2 \geq 0$$

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \end{pmatrix} \quad \mathbf{c}^{\mathrm{T}} = \begin{pmatrix} 2 \\ 3 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{A} = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 3 & 2 & 0 & 1 \end{pmatrix} \mathbf{b} = \begin{pmatrix} 8 \\ 12 \end{pmatrix}$$



$$\begin{pmatrix} \mathbf{A} & \mathbf{0} & \mathbf{b} \\ -\mathbf{c}^{\mathrm{T}} & f & 0 \end{pmatrix} =$$

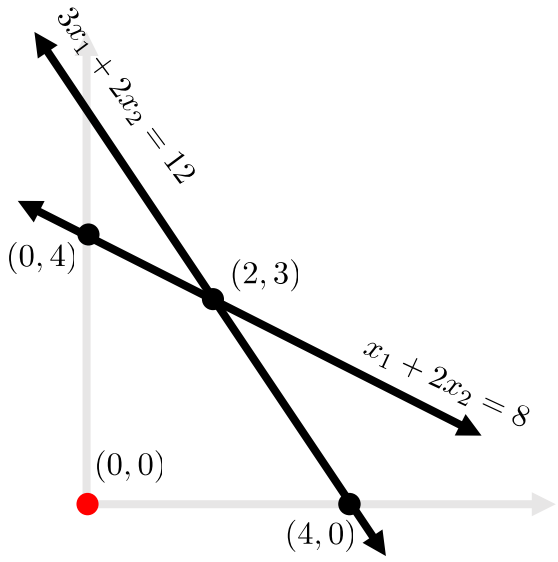
x_1	x_2	s_1	s_2	f	
1	2	1	0	0	8
3	2	0	1	0	12
-2	-3	0	0	1	0

basic (nonzero) variables

objective value

Step 2: While any variables have negative coefficients in the last row, choose the variable with the most negative.

x_1	x_2	s_1	s_2	f	
1	2	1	0	0	8
3	2	0	1	0	12
-2	-3	0	0	1	0



Why? Variables with nonzeros in the last row are not in the basic solution; they are zero in our current solution. The last row is our objective function:

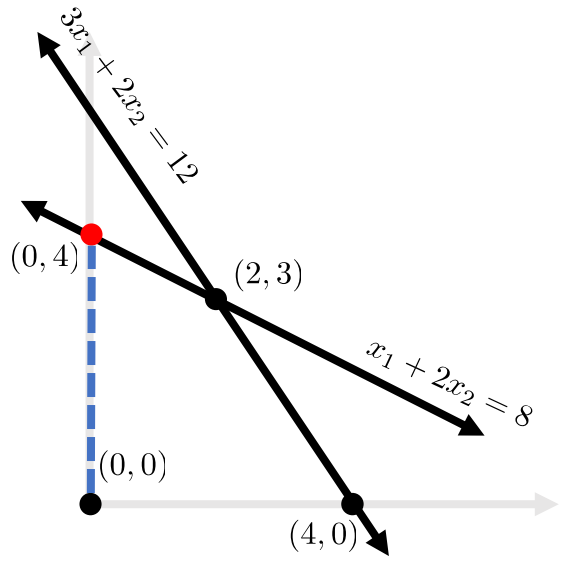
$$f = 2x_1 + 3x_2$$

To increase *f*, we want to make a zero variable nonzero by bringing it into. the basic solution. We choose the variable with the fastest change in *f*, i.e. the most negative in the tableau.

basic (nonzero) variables

objective value

Step 3: Bring the entering variable into the basic solution by zeroing out all rows except one. Which one? The row with the smallest ratio of the RHS to the coefficient of the entering variable.



x_1	x_2	s_1	s_2	f	
1	2	1	0	0	8
3	2	0	1	0	12
-2	-3	0	0	1	0

8 / 2 = 4 ← choose this row

12 / 2 = 6

$\xrightarrow{\frac{1}{2}R_1}$

x_1	x_2	s_1	s_2	f	
1/2	1	1/2	0	0	4
3	2	0	1	0	12
-2	-3	0	0	1	0

$\xrightarrow{R_2 - 2R_1}$

x_1	x_2	s_1	s_2	f	
1/2	1	1/2	0	0	4
2	0	-1	1	0	4
-2	-3	0	0	1	0

$\xrightarrow{R_3 + 3R_1}$

x_1	x_2	s_1	s_2	f	
1/2	1	1/2	0	0	4
2	0	-1	1	0	4
-1/2	0	3/2	0	1	12

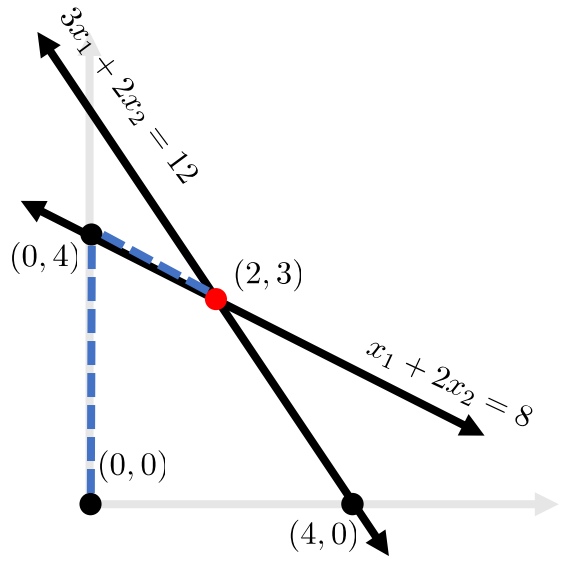
entering variable

basic (nonzero) variables

objective value

Step 2: While any variables have negative coefficients in the last row, choose the variable with the most negative.

Step 3: Bring the entering variable into the basic solution by zeroing out all rows except one. Which one? The row with the smallest ratio of the RHS to the coefficient of the entering variable.



x_1	x_2	s_1	s_2	f		
1/2	1	1/2	0	0	4	4 / 0.5 = 8
2	0	-1	1	0	4	4 / 2 = 2 ← choose this row
-1/2	0	3/2	0	1	12	

$\xrightarrow{\frac{1}{2}R_2}$

x_1	x_2	s_1	s_2	f	
1/2	1	1/2	0	0	4
1	0	-1/2	1/2	0	2
-1/2	0	3/2	0	1	12

$\xrightarrow{R_1 - \frac{1}{2}R_2}$

$\xrightarrow{R_3 + \frac{1}{2}R_2}$

x_1	x_2	s_1	s_2	f	
0	1	3/4	-1/4	0	3
1	0	-1/2	1/2	0	2
0	0	5/4	1/4	1	13

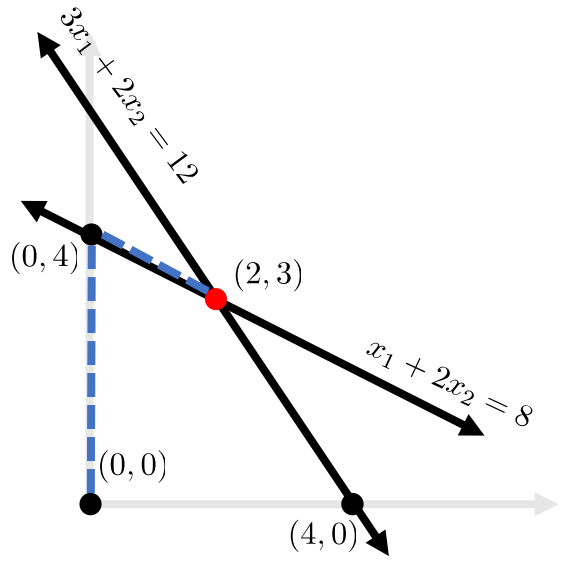
entering variable

basic (nonzero) variables

objective value

Step 2: While any variables have negative coefficients in the last row, choose the variable with the most negative.

Since there are no more negative coefficients in the last row, we have found the optimal solution! Any change would not improve the objective.



x_1	x_2	s_1	s_2	f	
0	1	3/4	-1/4	0	3
1	0	-1/2	1/2	0	2
0	0	5/4	1/4	1	13

The final solution can be read from the tableau:

$x_1 = 2$
$x_2 = 3$
$s_1, s_2 = 0$ (not in the basic solution)
$f = 13$

entering variable

basic (nonzero) variables

objective value